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MIT 6.859 – Randomness and Computation, with Ronitt Rubinfeld
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Problem Set 4 – Solutions

Problem 1

(a) In the following arguments, the given equalities hold for all $i \in [n]$, and therefore the eigenvactors are the same:

- The i-th eigenvalue of αP is $\alpha\lambda_i$, since:

$$(\alpha P)v_i = \alpha(Pv_i) = \alpha(\lambda_i v_i) = (\alpha\lambda_i)v_i$$

- The i-th eigenvalue of $P + I$ is $\lambda_i + 1$, since:

$$(P + I)v_i = Pv_i + Iv_I = \lambda_i v_i + v_i = (\lambda_i + 1)v_i$$

- Inductively:

$$P^k v_i = P(P^{k-1}v_i) = P\lambda_i^{k-1}v_i = \lambda_i^{k-1}(Pv_i) = \lambda_i^{k-1}\lambda_i v_i = \lambda_i^k v_i$$

  the i-th eigenvalue of P^k is λ_i^k.

- Applying the previous three results, it is straightforward that the i-the eigenvalue of $((P + I)/2)^k$ is:

$$\left(\frac{\lambda_i + 1}{2}\right)^k$$

(b) Let A be row-stochastic with its row vectors being $a_1, \ldots, a_n$. and let v and λ

be an eigenvector and its eigenvalue. Denote the i-th entry of v by v_i. Then:

$$
\begin{aligned}
|\lambda|\|v\|_1 &= \|\lambda v\|_1 \\
&= \|vA\|_1 \\
&= \|v_1 a_1 + \cdots + v_n a_n\|_1 \\
&\leq \|v_1 a_1\| + \cdots + \|v_n a_n\|_1, \quad \text{(triangular inequality)} \\
&= |v_1|\|a_1\|_1 + \cdots + |v_n|\|a_n\|_1 \\
&= |v_1| + \cdots + |v_n|, \quad \text{(stochasticity)} \\
&= \|v\|_1
\end{aligned}
$$

And therefore $|\lambda| \leq 1$.

(c) If we let V be the matrix whose columns are the v_i's, and α be a column vector consisting of the α_i's, we get that $w = V\alpha$. Using that $V^T V = 1$ for orthonormal matrices:

$$\|w\|^2 = w^T w = (V\alpha)^T (V\alpha) = \alpha^T (V^T V)\alpha = \alpha^T \alpha = \sum_{i=1}^{n} \alpha_i^2 = \|\alpha\|^2$$

Problem 2

For an arbitrary set R, let $\partial R = \{v \in V \mid \exists\ r \in R \wedge (r, v) \in E\}$. Let S_1 be an arbitrary subset of W_1 of size $n/2$. By definition, $|\partial S_1| \geq n/2 + \alpha n$, and therefore by the pigeonhole principle:

$$|\partial S_1 \cap W_2| \geq n/2 + \alpha n + n - \alpha n - n = n/2$$

Set S_2 to be an arbitrary subset of $\partial S_1 \cap W_2$ of size $n/2$ and repeat the above argument. Finally, the desired path is constructed backwards. Choose an arbitrary $v_k \in S_k$. Then by construction there exists at least one $v_{k-1} \in S_{k-1}$ such that $(v_{k-1}, v_k) \in E$. Continue in the same manner until all $v_1, \ldots, v_k$ are chosen.

Problem 3

(a) Use Cauchy-Schwarz in the following form:

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq n\left(\sum_{i=1}^{n} a_i^2\right)$$

To get:

$$\|\pi\|^2 = \sum_{x \in S(\pi)} \pi_x^2 \geq \frac{1}{n}\left(\sum_{x \in S(\pi)} \pi_x\right)^2 = \frac{1}{n}$$

(b) Simply:

$$\begin{aligned}
\|\pi - u\|^2 + 1/n &= \sum_{i=1}^{n} (\pi_i - 1/n)^2 + 1/n \\
&= \sum_{i=1}^{n} \left(\pi_i^2 - 2\pi_i/n + 1/n^2\right) + 1/n \\
&= \|\pi\|^2 - 2/n + n/n^2 + 1/n \\
&= \|\pi\|^2
\end{aligned}$$

(c) Let $R \subseteq V$ be any subset with $|R| \leq \alpha n$. And let ω be a distribution on V which is uniform on R and zero elsewhere. By construction, we have it that $|S(\omega P)|$ equals the size of R's neighbourhood including R itself.

Let P be the transition matrix of the random walk, and we will have to assume that G is regular, so that we know what is the stationary distribution π. Also $\|\cdot\|$ will denote the l_2 norm.

As in lecture, let $v_1, \ldots, v_n$ be an orthonormal set of eigenvectors for P. Then let $\omega = \sum_{i=1}^{n} \beta_i v_i$, and as noted in class $\pi = \beta_1 v_1$, and $\lambda_1 = 1$ (the first eigenvalue of P). Then we have that:

$$
\begin{aligned}
\frac{1}{|S(\omega P)|} &\le \|\omega P\|^2 && \text{from part (a)} \\
&= \|\omega P - \pi\|^2 + 1/n && \text{from part (b)} \\
&= (\lambda_1 \beta_1 - \beta_1)^2 + \left(\sum_{i=2}^{n} \lambda_i \beta_i \right)^2 + 1/n \\
&\le \lambda^2 \left(\sum_{i=2}^{n} \beta_i^2 \right) + 1/n \\
&= \lambda^2 \|\omega - \pi\|^2 + 1/n
\end{aligned}
$$

On the other hand, using that π is uniform due to G's regularity, we also have:

$$
\begin{aligned}
|R| \cdot \|\omega - \pi\|^2 &= |R| \left(|R| \left(\frac{1}{|R|} - \frac{1}{n} \right)^2 + \frac{1}{n^2} (n - |R|) \right) \\
&= 1 - \frac{|R|}{n}
\end{aligned}
$$

Then we apply the above two inequalities to get a bound on the expansion of R:

$$
\begin{aligned}
A &= \frac{|S(\omega P)|}{|R|} \\
&\ge \frac{1}{|R| \left(\lambda^2 \|\omega - \pi\|^2 + 1/n \right)} \\
&= \frac{1}{|R| \lambda^2 \|\omega - \pi\|^2 + |R|/n} \\
&= \frac{1}{\lambda^2 (1 - |R|/n) + |R|/n} \\
&\ge \frac{1}{\lambda^2 (1 - \alpha) + \alpha}
\end{aligned}
$$

The last inequality follows from the fact that $|R|/n \le \alpha$ implies $\lambda^2(1 - |R|/n) + |R|/n \le \lambda^2(1 - \alpha) + \alpha$ since $\lambda^2 < 1$.