Problem 1

(a) In the following arguments, the given equalities hold for all $i \in [n]$, and therefore the eigenvectors are the same:

- The $i$-th eigenvalue of $\alpha P$ is $\alpha \lambda_i$, since:
  \[ (\alpha P)v_i = \alpha (Pv_i) = \alpha (\lambda_i v_i) = (\alpha \lambda_i) v_i \]

- The $i$-th eigenvalue of $P + I$ is $\lambda_i + 1$, since:
  \[ (P + I)v_i = Pv_i + Iv_i = \lambda_i v_i + v_i = (\lambda_i + 1) v_i \]

- Inductively:
  \[ P^k v_i = P(P^{k-1}v_i) = P\lambda_i^{k-1}v_i = \lambda_i^{k-1}(Pv_i) = \lambda_i^{k-1}\lambda_i v_i = \lambda_i^k v_i \]
  the $i$-th eigenvalue of $P^k$ is $\lambda_i^k$.

- Applying the previous three results, it is straightforward that the $i$-the eigenvalue of $((P + I)/2)^k$ is:
  \[ \left( \frac{\lambda_i + 1}{2} \right)^k \]

(b) Let $A$ be row-stochastic with its row vectors being $a_1, \ldots, a_n$. and let $v$ and $\lambda$
be an eigenvector and its eigenvalue. Denote the $i$-th entry of $v$ by $v_i$. Then:

$$|\lambda||v||_1 = \|\lambda v\|_1$$

$$= \|vA\|_1$$

$$= \|v_1a_1 + \cdots + v_na_n\|_1$$

$$\leq \|v_1a_1\| + \cdots + \|v_na_n\|_1$$ (triangular inequality)

$$= |v_1||a_1||_1 + \cdots + |v_n||a_n||_1$$

$$= |v_1| + \cdots + |v_n|$$ (stochasticity)

$$= \|v||_1$$

And therefore $|\lambda| \leq 1$.

(c) If we let $V$ be the matrix whose columns are the $v_i$’s, and $\alpha$ be a column vector consisting of the $\alpha_i$’s, we get that $w = V\alpha$. Using that $V^TV = 1$ for orthonormal matrices:

$$\|w\|^2 = w^Tw = (V\alpha)^T(V\alpha) = \alpha^T(V^TV)\alpha = \alpha^T\alpha = \sum_{i=1}^{n} \alpha_i^2 = \|\alpha\|^2$$
Problem 2

For an arbitrary set $R$, let $\partial R = \{v \in V \mid \exists \ r \in R \land (r, v) \in E\}$. Let $S_1$ be an arbitrary subset of $W_1$ of size $n/2$. By definition, $|\partial S_1| \geq n/2 + \alpha n$, and therefore by the pigeonhole principle:

$$|\partial S_1 \cap W_2| \geq n/2 + \alpha n + n - \alpha n - n = n/2$$

Set $S_2$ to be an arbitrary subset of $\partial S_1 \cap W_2$ of size $n/2$ and repeat the above argument. Finally, the desired path is constructed backwards. Choose an arbitrary $v_k \in S_k$. Then by construction there exists at least one $v_{k-1} \in S_{k-1}$ such that $(v_{k-1}, v_k) \in E$. Continue in the same manner until all $v_1, \ldots, v_k$ are chosen.
Problem 3

(a) Use Cauchy-Schwarz in the following form:

\[
\left( \sum_{i=1}^{n} a_i \right)^2 \leq n \left( \sum_{i=1}^{n} a_i^2 \right)
\]

To get:

\[
\| \pi \|^2 = \sum_{x \in S(\pi)} \pi_x^2 \geq \frac{1}{n} \left( \sum_{x \in S(\pi)} \pi_x \right)^2 = \frac{1}{n}
\]

(b) Simply:

\[
\| \pi - u \|^2 + 1/n = \sum_{i=1}^{n} (\pi_i - 1/n)^2 + 1/n
\]

\[
= \sum_{i=1}^{n} (\pi_i^2 - 2\pi_i/n + 1/n^2) + 1/n
\]

\[
= \| \pi \|^2 - 2/n + n/n^2 + 1/n
\]

\[
= \| \pi \|^2
\]

(c) Let \( R \subseteq V \) be any subset with \(|R| \leq \alpha n\). And let \( \omega \) be a distribution on \( V \) which is uniform on \( R \) and zero elsewhere. By construction, we have it that \(|S(\omega P)|\) equals the size of \( R \)'s neighbourhood including \( R \) itself.

Let \( P \) be the transition matrix of the random walk, and we will have to assume that \( G \) is regular, so that we know what is the stationary distribution \( \pi \). Also \( \| \cdot \|\) will denote the \( l_2 \) norm.

As in lecture, let \( v_1, \ldots, v_n \) be an orthonormal set of eigenvectors for \( P \). Then let \( \omega = \sum_{i=1}^{n} \beta_i v_i \), and as noted in class \( \pi = \beta_1 v_1 \), and \( \lambda_1 = 1 \) (the first eigenvalue of \( P \)). Then we have that:
\[
\frac{1}{|S(\omega P)|} \leq \|\omega P\|^2 \quad \text{from part (a)}
\]
\[
= \|\omega P - \pi\|^2 + 1/n \quad \text{from part (b)}
\]
\[
= (\lambda_1 \beta_1 - \beta_1)^2 + \left( \sum_{i=2}^{n} \lambda_i \beta_i \right)^2 + 1/n
\]
\[
\leq \lambda^2 \left( \sum_{i=2}^{n} \beta_i^2 \right) + 1/n
\]
\[
= \lambda^2 \|\omega - \pi\|^2 + 1/n
\]

On the other hand, using that \( \pi \) is uniform due to \( G \)'s regularity, we also have:
\[
|R| \cdot \|\omega - \pi\|^2 = |R| \left( |R| \left( \frac{1}{|R|} - \frac{1}{n} \right)^2 + \frac{1}{n} (n - |R|) \right)
\]
\[
= 1 - \frac{|R|}{n}
\]

Then we apply the above two inequalities to get a bound on the expansion of \( R \):
\[
A = \frac{|S(\omega P)|}{|R|}
\]
\[
\geq \frac{1}{|R| (\lambda^2 \|\omega - \pi\|^2 + 1/n)}
\]
\[
= \frac{1}{|R| \lambda^2 \|\omega - \pi\|^2 + |R|/n}
\]
\[
= \frac{1}{\lambda^2 (1 - |R|/n) + |R|/n}
\]
\[
\geq \frac{1}{\lambda^2 (1 - \alpha) + \alpha}
\]

The last inequality follows from the fact that \( |R|/n \leq \alpha \) implies \( \lambda^2 (1 - |R|/n) + |R|/n \leq \lambda^2 (1 - \alpha) + \alpha \) since \( \lambda^2 < 1 \).