9/28 Ben Green

Behrend's Example (1946) $\exists A \subseteq [N]$, $\frac{|A|}{N} \gg e^{-c\sqrt{\log N}}$, A contains no 3AP.

Pf: Let $S \subseteq \{1..d\}^d$, $|S| \geq \frac{1}{d} L^{d2}$, all points of S lie on sphere..

By convexity, S contains no 3 point on a line.

Consider the image $A = \psi(S)$ where $\psi : \mathbb{Z}^d \to \mathbb{Z}$

is the map $(x_1 .. x_d) \mapsto x_1 + (2L)x_2 + \cdots + (2L)^{d-1} x_d$.

(Exercise, use no carries. Can be formalized.)

[We'll see later that $\psi|_{\{1..d\}^{2d}}$ is an example of

a Freiman-isomorphism; a map $\varphi : B \to B'$ s.t.

$$x_1 + x_2 = x_3 + x_4 \iff \varphi(x_1) + \varphi(x_2) = \varphi(x_3) + \varphi(x_4)$$

How big can we make A? Roughly, $N \sim (2L)^d$,

$\Downarrow$

$\log N \sim d \log L$

$$\frac{|A|}{N} = \frac{1}{d 2^d L^2} \approx \frac{1}{2^d L^2}$$

$$\approx e^{-\left(\frac{\log N}{\log L} + \log L\right)} \text{ (sloppy...)}.$$

Digression: What is the largest set $A \subseteq (\mathbb{Z}/3\mathbb{Z})^n$ with no 3-term AP?

Analog of Roth's argument $\Rightarrow$ $|A| \leq \frac{3^n}{n}$ ~~[illegible]~~ $\frac{3^n}{n^c}$

Best known lower bound is $\exists A \ |A| \geq (2.6..)^n$ (by boring example))?

Easy ex., regression: $AP_3(f_1, f_2, f_3) = \sum_r \hat{f_1}(r)\hat{f_2}(r)\hat{f_3}(r)$.

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§2. Ruzsa Calculus (calculus of subsets).

Suppose A, B are subset of some ambient abelian group.

$A+B := \{a+b : a \in A, b \in B\}$

if $k, l \geq 0$ integers, $kA - lB = \{a_1 + \dots + a_k - b_1 - \dots - b_l : a_i \in A, b_j \in B\}$

"doubling constant of A" $= \sigma[A] := \frac{|A+A|}{|A|}$.

Theorem: (Ruzsa's Triangle inequality).

Let U, V, W be finite subsets of an abelian group.
Then $|U| \cdot |V-W| \leq |U-V| \cdot |U-W|$.

Remark: If we define the "Ruzsa's distance"
$$d(U,V) := \log \frac{|U-V|}{|U|^{1/2} \cdot |V|^{1/2}},$$
then is $d(U,W) \leq d(U,V) + d(V,W)$

Proof: For ~~ea~~ each $d \in V-W$ select (arbitrarily) $v(d), w(d)$ such that $d = v(d) - w(d)$.

Define $\psi : U \times (V-W) \longrightarrow (U-V) \times (U-W)$ by $\psi(u,d) = \big(u - v(d), u - w(d)\big)$. Note that ψ is injective:

$\psi(u,d) = \psi(u',d') \Rightarrow$

$$d = (u - v(d)) - (u - w(d))$$

$$= (u' - v(d')) - (u' - w(d')) = d'$$

Hence $u = u'$ as well. ∎

Theorem (The 2nd Rusza Inequality).

$$d(U,-V) \leq 3\, d(U,V).$$

Pf: Define $r(x) := \#\text{pairs } \{(u,v) : u \in U, v \in V, u - v = x\}$

$s(x) := \#\{(u,v) : u \in U, v \in V, u + v = x\}$

Note $\sum_x r(x) = \sum_x s(x) = |U| \cdot |V|$.

and $\sum_x r(x)^2 = \sum_x s(x)^2 = \#\{(u_1, v_1, u_2, v_2) : u_1 + v_1 = u_2 + v_2\}$.

But by Cauchy-Schwarz, $\sum r(x)^2 \geq \frac{1}{|U-V|}\left(\sum_x r(x)\right)^2 \geq \frac{|U|^2 |V|^2}{|U-V|}$

(Note $r(x)$ is supported on $U-V$.)

Hence $\sum s(x)^2 \geq \dfrac{|U|^2|V|^2}{|U-V|}$. Hence since $\sum_x s(x) = |U|\cdot|V|$,

there is at least one X such that $s(x) \geq \dfrac{|U|\cdot|V|}{|U-V|}$. Fix this x.

Let $S \subseteq U \times V$ be the set of pairs (u,v) with $u+v = x$.

I shall define an injective map $\varphi: S \times (U+V) \longrightarrow (U-V) \times (U-V)$.

If this is done, then $|S|\cdot|U+V| \leq |U-V|^2$

$$\Rightarrow |U+V| \leq \frac{|U-V|^3}{|U|\cdot|V|}. \text{ QED.}$$

If $w \in U+V$, pick $u(w), v(w)$ arbitrarily so that

$u(w) + v(w) = w$.

Define $\varphi((u,v), w) = (u - v(w), u(w) - v)$.

Why is this injective?

Assume $\varphi((u,v),w) = \varphi((u',v'),w')$

We have $w = u(w) + v(w) = u+v-(u-v(w)) + (u(w)-v)$

$= x - (u-v(w)) + (u(w)-v)$

$= u'+v' - (u'-v(w')) + (u(w')-v')$

$= w'$!

Easy to show that $u=u'$, $v=v'$! ∎

Ruzsa calculus. Let $K \geq 1$ be an "approximation parameter", fixed throughout any Ruzsa calculus argument.

If X, Y are non-negative reals, write $X \lesssim Y$ to mean $X \leq CK^C Y$.
"X ~~and~~ bounded by Y ~~are the~~ up to polynomials in K".

Write $X \approx Y$ to mean $X \lesssim Y$ and $Y \lesssim X$.

Let U, V be sets in some ambient abelian group.

Write $U \sim V$ to mean $\frac{|U-V|}{|U|^{1/2}|V|^{1/2}} \approx 1$.

If $\sigma[U] \approx 1$ we say that U is an "approximate group".

"Ruzsa calculus" Let $K \geq 1$ be a parameter for approximate notation.

U, V, W sets in some ambient abelian group.

Then (i) If $U \sim V$ and $V \sim W$ then $U \sim W$.

(ii) If $U \sim V$ then $U \sim -V$.

(iii) Suppose $U \sim V$ and that there is some X such that $\sigma[W] = 1$

$$|U \cap (x+W)| \approx |U| \approx |W|.$$ Then $U \sim W$.

(iii)' Suppose $\sigma[U], \sigma[W] \approx 1$ and there is an x such that

$$|U \cap (x+W)| \approx |U| \approx |W|. \text{ Then } U \approx W.$$

Pf:

(i) $\Leftarrow$ Ruzsa's Δ inequality

(ii) $\Leftarrow$ Ruzsa's 2nd inequality

(iii) $\Leftarrow$ Exercise: hint $|U \cap W| \cdot |U - W| \leq |U - U| \cdot |W - W|$ by Ruzsa's Δ inequality.

Since $(U \cap W) - U \subseteq U - U$

$(U \cap W) - W \subseteq W - W$.