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Prop: Let $G = \mathbb{Z}/N'\mathbb{Z}$ and $f : G \to \mathbb{C}$ is 1-bounded such that there is $S \subseteq G$, $|S| \geq \delta |G|$ and $\varphi : S \to \widehat{G}$ such that $|\Delta(f;h)^\wedge(\varphi(h))| \geq \eta$. Write $\Gamma = \{(h, \varphi(h)) : h \in S\}$.

Then $\omega(\Gamma) \geq \eta \delta^8$.

Pf: Hypothesis implies $\mathbb{E}_h 1_S(h) |\Delta(f;h)^\wedge(\varphi(h))^2| \geq \delta \eta^2$.

Expanding out, obtain:

$$\left| \mathbb{E}_h 1_S(h) \mathbb{E}_{x,y} f(x)\overline{f(x+h)f(y)}f(y+h) e\left(\frac{\varphi(h)(x-y)}{N'}\right) \right| \geq \delta\eta^2.$$

Substitute $y = x+k$ gives:

$$\left| \mathbb{E}_{h,x,k} 1_S(h) f(x) \overline{f(x+h) f(x+k)} f(x+h+k)\, e\left(\frac{\varphi(h)k}{N'}\right) \right| \geq \delta\eta^2.$$

By triangle inequality:

$$\mathbb{E}_{x,k} \left| \mathbb{E}_h 1_S(h) f(x+h) f(x+h+k) e\left(\frac{-\varphi(h)k}{N'}\right) \right| \geq \delta\eta^2.$$

By Cauchy–Schwarz:

$$\mathbb{E}_{x,k} \left| \mathbb{E}_h 1_S(h) f(x+h) f(x+h+k) e\left(\frac{-\varphi(h)k}{N'}\right) \right|^2 \geq \delta^2 \eta^4$$

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This can, after a little thought, be written as:

$$\mathbb{E}_k \| F_k * G_k \|_2^2 \geq \delta^2 \eta^4, \quad \text{where} \begin{cases} G_k(t) = \Delta(f, k)(t) \\ F_k(t) = 1_S(t)\, e\left(\frac{-\varphi(t)k}{N'}\right) \end{cases}$$

In terms of Fourier transform on G, this implies

$$\mathbb{E}_k \sum_r |\hat{F}_k(r)|^2 |\hat{G}_k(r)|^2 \geq \delta^2 \eta^4 \qquad \left(\begin{array}{c}\text{F.T. of conv.} = \text{prod. of F.T.s} \\ + \text{Parseval}\end{array}\right)$$

By Cauchy-Schwarz and $\sum_r |\hat{G}_k(r)|^4 \leq 1$ (Parseval, applied to $G_k * G_k$)

this tells us that $\mathbb{E}_k \left(\sum_r |\hat{F}_k(r)|^4\right)^{1/2} \geq \delta^2 \eta^4$.

Cauchy-Schwarz again $\Rightarrow$ $\mathbb{E}_k \sum_r |\hat{F}_k(r)|^4 \geq \delta^4 \eta^8$.

What is the LHS? Expand out,

$$\mathbb{E}_k \sum_r \left| \mathbb{E}_t 1_S(t)\, e\left(\frac{-\varphi(t)k - rt}{N'}\right)\right|^4$$

expanding out using orthogonality relations gives

$$\mathbb{E}_{t_1,t_2,t_3,t_4} \mathbb{E}_k\, e\left(\frac{k}{N'}\left(\varphi(t_1) - \varphi(t_2) - \varphi(t_3) + \varphi(t_4)\right)\right) \cdot \sum_r e\left(\frac{r}{N'}(t_1 - t_2 - t_3 + t_4)\right)$$

$$= \frac{1}{(N')^3} \# \left\{ \text{solutions to } \begin{array}{l} t_1+t_2=t_3+t_4 \\ \varphi(t_1)+\varphi(t_2)=\varphi(t_3)+\varphi(t_4) \end{array} \right\}$$

This is equivalent to $\omega(\Gamma) \geqslant \delta\eta^8$. $\square$

$\|f\|_{U^3} \geqslant \delta \Rightarrow$ for at least $\frac{\delta^8}{2}N$ values of h we have:

$$|\Delta(f;h)^\wedge(\varphi(h))| \geqslant \frac{\delta^8}{2}.$$

<u>Theorem</u> (Local inverse theorem for the U^3 norm):

Suppose $f: G \to \mathbb{C}$ is 1-bounded and that $\|f\|_{U^3} \geqslant \delta$.

Then there is an A.P. $Q \subseteq G$, $|Q| \geqslant \exp(-\delta^{-C}) N^{\delta^C}$ together with quadratic ~~phases~~ polynomials $\psi_0, \dots, \psi_{N'-1}$ such that

$$\mathbb{E}_{i=0,\dots,N'-1} \left| \mathbb{E}_{x \in Q+i} f(x) e(\psi_i(x)) \right| \geqslant \delta^C.$$

<u>Remarks</u>: Called "local inverse theorem" because $|Q| \ll N$. Such a theorem is useless for handling most functions f which arise in prime number theory, e.g. $f = \mu =$ Möbius. Indeed handling sums $\sum_{x \in I} \mu(x)$ even without a quadratic phase is hopeless if $I \subseteq \{1, \dots, N\}$ and $|I| \ll N^{1-c}$.

What is needed for such applications is a "global" inverse theorem, in which f is shown to correlate with something on all of G. Unfortunately, quadratic phases are not enough.

We require "2-step nil sequences". (To be elaborated later.)

Proof: By Gowers proposition (just proved) the function φ (for which $|\Delta(f;h)^\wedge(\varphi(h))| \geq \frac{\delta^8}{2}$) is "weakly linear".

If φ were exactly linear, say $\varphi(h) = -2h$? We then have

$$\mathbb{E}_h |\Delta(f;h)^\wedge(-2h)|^2 \geq \delta^c.$$

Expanding out + substitution:

$$\mathbb{E}_h \mathbb{E}_{x,k} f(x)\overline{f(x+h)}\,\overline{f(x+k)} f(x+h+k)\, e\left(\frac{2hk}{N'}\right) \geq \delta^c$$

$$\Rightarrow \mathbb{E}_{h,x,k} \tilde{f}(x)\overline{\tilde{f}(x+h)}\,\overline{\tilde{f}(x+k)}\,\tilde{f}(x+h+k) \geq \delta^c \text{ where}$$

$$\tilde{f}(x) = f(x)\, e\left(\frac{x^2}{N'}\right).$$

$$\left(\Delta(f;h)(x) = f(x)\overline{f(x+h)}.\right)$$

$$\Rightarrow \|\tilde f\|_{U^2} \geq \delta^c \xrightarrow{U^2\text{-inverse thm}} \left|\mathbb{E}_x \tilde f(x) e(-\theta x)\right| \geq \delta^c \text{ for some } \theta$$

$$\Rightarrow \left|\mathbb{E}_x f(x) e\left(\frac{x^2}{N'} - \theta x\right)\right| \geq \delta^c, \text{ that is,}$$

f correlates with a quadratic phase.

We now sketch pf. of local inverse theorem:

<u>Step 0</u>: There are many ($\geq \delta N$) values of h such that $|\Delta(f;h)^\wedge(\varphi(h))| \geq \delta'$

<u>Step 1</u>: Apply proposition at start of lecture:
if $\Gamma = \{(h, \varphi(h))\}$ then $\omega(\Gamma) \geq \delta^c$.

<u>Step 2</u>: By Balog–Szemerédi–Gowers, there is $\Gamma' \subseteq \Gamma$, $|\Gamma'| \geq \delta^c |\Gamma|$.
such that $\sigma[\Gamma'] = \dfrac{|\Gamma' + \Gamma'|}{|\Gamma'|} \leq \delta^{-c}$.

<u>Step 3</u>**: Apply Freiman–Ruzsa to place Γ' inside a G.A.P., P,
of dimension $d = \delta^{-c}$ and size $\leq \exp\left(\frac{1}{\delta^c}\right)|\Gamma'|$.

<u>Step 4</u>: Foliate P into 1-dimensional (i.e. regular) arithmetic progressions of size $\gtrsim N^{1/d}$.

Γ' has big intersection with one of these, say Q. But if graph of a function is contained in a (1-dimensional) progression then that function is affine linear.

Step 5: We have $\mathbb{E}_{h \in Q} \left| \Delta(f;h)^\wedge(ah+b) \right|^2 \geq \delta^{c}$.

Step 6: Model the argument for when $\varphi(h) = -2h$, noting that Q is just a subprogression.

What were $*$ and $\dagger$ in Step 3?

($*$) Γ is a subset of $\mathbb{Z}/N'\mathbb{Z} \times \mathbb{Z}/N'\mathbb{Z}$ and we didn't prove Freiman–Ruzsa in this context. To get around, lift Γ' to subset of $\{1,..,N'\} \times \{1,..,N'\} \subseteq \mathbb{Z}^2$. This increases the doubling by 4, maybe. Now project to $\mathbb{Z}$ via $(x,y) \mapsto x + My$ for some huge M (say $10N'$).

($\dagger$)

$$\begin{tikzcd}
 & \text{Freiman Ruzsa} \arrow[dr, "\text{Step } 3^\dagger"] & \\
\text{Bogolyubov's Lemma} \arrow[ur, "\text{Chang's Covering Argument}"] \arrow[rr, "\text{quicker}"'] & & \bullet
\end{tikzcd}$$

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