

10/6 Ben Green

Theorem (Gowers): Suppose $A \subseteq \{1, \dots, N\}$, $|A| \geq N (\log \log N)^{-c}$ for some absolute $c > 0$. Then A contains a 4-term AP.

Remark: Szemerédi showed 40 years ago that if $|A| \geq N \cdot \omega(N)$ for some function $\omega(N) \rightarrow 0$. Szemerédi's function very slowly decaying.

Gowers gave the first "reasonable" bound.

The initial stages of proof model pf. of Roth's than closely. In particular, we employ "density increment" based on:

Prop: Let $P \subseteq \mathbb{Z}$ be a progression of length N , and suppose $A \cap P$ has size αN .

Then at least one is true:

(i) A has at least $\frac{1}{20} \alpha^4 N^2$ 4-term progressions.

(ii) A has density at least $\alpha + \alpha^c$ on some subprogression $P' \subseteq P$,

$$|P'| \geq \exp(-\alpha^{-c}) N \alpha^c$$

Exercise: Convince yourself that iteration of prop. gives bound!

Pf: Introduce prime $N' \sim 10N$. We'll work in $G = \mathbb{Z}/N'\mathbb{Z}$.

Suppose $f_1, f_2, f_3, f_4: G \rightarrow \mathbb{C}$. Define:

$$AP_4(f_1, r, f_3, f_4) = \mathbb{E}_{x \in G} f_1(x) f_2(x+r) f_3(x+2r) f_4(x+3r)$$

①

Wlog assume $P = [N]$.
 Set $f = \frac{1}{A} - \alpha \mathbb{1}_{[N]}$ "balanced function of A ".

Exactly as in proof of Roth, obtain:

Lemma: Suppose $A \subseteq [N]$, $|A| = \alpha N$. Suppose A contains fewer than $\frac{1}{20} \alpha^4 N^2$ 4-term APs. Then there are L -bounded functions $g_1, \dots, g_4: G \rightarrow \mathbb{C}$, ~~the~~ ^{at} ~~least~~ ^{one} of which is f , such that:

$$|AP_4(g_1, g_2, g_3, g_4)| \geq c \alpha^4.$$

Proposition (Generalized von Neumann Theorem): Suppose $g_1, \dots, g_4: G \rightarrow \mathbb{C}$ are L -bounded. Then $|AP_4(g_1, \dots, g_4)| \leq \|g_i\|_{U^3}$ for each $i=1,2,3,4$.

(We'll illustrate case $i=1$)

PF: Note that $AP_4(g_1, g_2, g_3, g_4) = \mathbb{E}_{x, y, z \in G} g_1(x+y+z) g_2(\frac{1}{2}y + \frac{2}{3}z) g_3(-x + \frac{1}{3}z) g_4(-2x - \frac{1}{2}y)$

~~Cauchy-Schwarz~~ Thus $|AP_4(\dots)| \leq \mathbb{E}_{y, z \in G} \left| \mathbb{E}_x (g_1(x+y+z) g_3(-x + \frac{1}{3}z) g_4(-2x - \frac{1}{2}y)) \right|$

$$\stackrel{\text{(Cauchy)}}{\leq} \left(\mathbb{E}_{y, z \in G} \left| \dots \right|^2 \right)^{1/2}$$

Do another two Cauchy-Schwarz to eliminate g_3, g_4 . We'll end up with

$$|AP_4| \leq \left(\mathbb{E}_{\substack{x, y, z \\ x', y', z'}} \overline{g_1(x+y+z)} g_1(x'+y'+z') \dots \right)^{1/8} = \|g_1\|_{U^3} \quad \square$$

(1)

Summary so far: If $A \subseteq \{1, \dots, N\}$, $|A| = \alpha N$ and if A has fewer than $\frac{1}{20} \alpha^4 N^2$

4-term APs, then $\|f\|_{U^3} \geq c \alpha^4$ where f is the balanced function of A .

To conclude pf. of density increment prop. (and hence of Gowers theorem), suffices:

Theorem (weak inverse thm for U^3 norm):

Suppose $\|f\|_{U^3} \geq \delta$, that f is $\sqrt{\text{real-valued}}$ 1-bounded and that $\mathbb{E}f = 0$.
Then there is a progression $P \subseteq \{1, \dots, N\}$, $|P| \geq \exp(-\delta^{-c}) N^{\delta^c}$ such that
 $\mathbb{E}_{x \in P} f(x) \geq \delta^c$.

Pf: First steps: Observe that $\|f\|_{U^3}^8 = \mathbb{E}_{h \in G} \|\Delta(f; h)\|_{U^2}^4$

where $\Delta(f; h)(x) = f(x)f(x+h)$.

If $\|f\|_{U^3} \geq \delta$ then there are $\geq \frac{\delta^8}{2} N$ values of h such that

$$\|\Delta(f; h)\|_{U^2} \geq \frac{\delta^2}{2}.$$

By the inverse theorem for U^2 , for each such h there is some

$q(h) \in \mathbb{Z}/N\mathbb{Z}$ such that $|\Delta(f; h)^{\wedge}(q(h))| \geq \frac{\delta^4}{4}$. (cf. Proof of U^2 inverse thm.)

Remark: When is $\|f\|_{U^3} = 1$? Easy to see $|f(x)| = 1$, for all x ,

$$\text{hence } f(x) = e^{2\pi i \varphi(x)} = e(\varphi(x)).$$

$$\|f\|_{U^3} = \mathbb{E}_{x, h_1, h_2, h_3} e(\varphi(x) - \varphi(x+h_1) - \varphi(x+h_2) - \varphi(x+h_1+h_2+h_3))$$

If this equals 1 then $\varphi(x) - \varphi(x+h_1) - \dots - \varphi(x+h_1+h_2+h_3) = 0 \pmod{1}$
for all x, h_1, h_2, h_3 . $\iff \varphi(x) = ax^2 + bx + c \pmod{1}$.

So, in some sense U_3 norms captures some notion of being almost quadratic.

$$\text{Suppose that } f(x) = e\left(\frac{x^2}{N'}\right). \text{ then } \Delta(f; h)(x) = e\left(\frac{x^2}{N'} - \frac{(x+h)^2}{N'}\right) = e\left(\frac{-2xh}{N'}\right) = e\left(-\frac{h^2}{N'}\right).$$

$$\text{So } |\Delta(f; h)^\wedge(-2h)| = 1 \text{ and } |\Delta(f; h)^\wedge(r)| = 0 \text{ else.}$$

$$\text{So in this example } \varphi(h) = -2h.$$

Proposition (Gowers). Suppose $f: G \rightarrow \mathbb{C}$ is 1-bounded

and there is $S \subseteq G$, $|S| = \delta|G|$ such that for all $h \in S$ there is a $\varphi(h) \in G$ such that $|\Delta(f; h)^\wedge(\varphi(h))| \geq \eta$. Then φ is

"weakly linear" in the sense:

Writing $\Gamma = \{(h, \varphi(h)) : h \in S\} \subseteq G \times G$, we have $\omega(\Gamma) \geq \delta \eta^8$. ↙ normalized additive energy.

1 $\omega(\Gamma)$ ~ solutions to $(h_1, \varphi(h_1)) + (h_2, \varphi(h_2)) =$
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3 $= (h_3, \varphi(h_3)) + (h_4, \varphi(h_4))$
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5 i.e. $h_1 + h_2 = h_3 + h_4$ and
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7 $\varphi(h_1) + \varphi(h_2) = \varphi(h_3) + \varphi(h_4)$
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