

Bourgain-Glibichuk

$$H \subseteq \mathbb{F}_p^* \setminus \{1\}, |H| = p^\delta \text{ then } \sup_{\xi \neq 0} \frac{1}{|H|} \left| \sum_{x \in H} e\left(\frac{x\xi}{p}\right) \right| \leq p^{-\delta'}, \quad \delta' = \delta'(\delta) > 0$$

Recall, if $H \subseteq \mathbb{F}_p$ is any set, write $\text{Spec}_2(H) := \left\{ \xi \neq 0 \mid \left| \sum_{x \in H} e\left(\frac{x\xi}{p}\right) \right| \geq \alpha |H| \right\}$

Lemma: Suppose $B \subseteq \text{Spec}_2(H)$, then for a proportion at least $\frac{\alpha^2}{2}$ of the pairs $x, y \in B$ we have $x, y \in \text{Spec}_{\alpha/2}(H)$.

Pf: Suppose $B = \{\xi_1, \dots, \xi_k\}$. For each j choose a complex $\# g_j$ with $|g_j| = 1$, such that $g_j \sum_{x \in H} e\left(\frac{x\xi_j}{p}\right)$ is real and at least $\alpha |H|$.

Summing over j and swapping summation order:

$$\sum_{x \in H} \sum_{j=1}^k g_j e\left(\frac{\xi_j x}{p}\right) \geq k\alpha |H|$$

Applying CS gives:

$$\sum_{x \in H} \sum_{k < j, j' < k} c_j \bar{c}_{j'} e\left(\frac{(\xi_j - \xi_{j'})x}{p}\right) \geq k^2 \alpha^2 |H|. \quad \Rightarrow$$

$$\Rightarrow \sum_{j, j'} \left| \sum_{x \in H} e\left(\frac{(\xi_j - \xi_{j'})x}{p}\right) \right| \geq k^2 \alpha^2 |H|. \quad \text{It follows easily that}$$

$$\text{for at least } \frac{\alpha^2 k^2}{2} \text{ pairs } j, j' \text{ we have } \left| \sum_{x \in H} e\left(\frac{(\xi_j - \xi_{j'})x}{p}\right) \right| \geq \frac{\alpha^2 |H|}{2}. \quad \square$$

Additive-Multiplicative BSG Theorem

Suppose $A \subseteq \mathbb{F}_p$ and $B \subseteq \mathbb{F}_p^*$ such that $\omega(A, bA) \approx 1$ for all $b \in B$.

Then there are sets $A' \subseteq A$, $B' \subseteq B$ (for some $x \neq 0$) such that

$$\delta[A', bA'] \approx 1 \text{ for all } b \in B' \text{ and } \begin{cases} |A'| \approx |A| \\ |B'| \approx |B|. \end{cases}$$

pf: By the usual BSG, there are, for each $b \in B$, subsets $X_b \subseteq A$ and $Y_b \subseteq A$ such that $|X_b| \approx |Y_b| \approx |A|$ and $\delta(X_b, b \cdot Y_b) \approx 1$.

The aim is to remove dependence on b .

Lemma: Suppose S is a finite set and S_1, \dots, S_k are subsets of S with $|S_i| \geq \delta |S|$. Then there is an i such that

$$|S_i \cap S_j| \geq \frac{\delta^2}{2} |S| \text{ for at least } \frac{\delta^2 k}{2} \text{ values of } j.$$

Proof: Cauchy-Schwarz: exercise. \square

Apply this lemma to sets $(X_b \times Y_b)_{b \in B}$ which are subsets of $A \times A$ of size $\approx |A \times A|$. So $\delta \approx 1$. I get a $b_0 \in B$ such that

$|X_{b_0} \cap X_b|, |Y_{b_0} \cap Y_b| \approx |A|$ for a set B' of b 's with $|B'| \approx |B|$.

I also know that $X_b \sim b \cdot Y_b$ and $X'_b \sim b_0 \cdot Y_{b_0}$ (where recall $u \sim v \Leftrightarrow \delta[uv] \approx 1$).

From Ruzsa Calculus we have: $\delta[X_b], \delta[X_{b_0}], \delta[Y_b], \delta[Y_{b_0}] \approx 1$.

Also, by statement (iii) in Ruzsa calculus, we have $X_{b_0} \sim X_b$ and $Y_{b_0} \sim Y_b$.

$$\text{Also, } X_{b_0} \sim X_b \sim b \cdot Y_b \xrightarrow{\quad\quad\quad} b Y_{b_0} \sim b Y_b$$

$$\text{Hence } b Y_{b_0} \sim b Y_b \Rightarrow Y_{b_0} \sim \frac{b}{b_0} \cdot Y_{b_0}$$

Taking $A' = Y_{b_0}$ and redefining B' to be this set of $\frac{b}{b_0}$'s, we

conclude the result. \blacksquare

Lemma: Let $H \leq \mathbb{F}_p^*$ be a multiplicative subgroup. Let $0 \leq \alpha \leq 1$,

let $A = \text{Spec}_\alpha(H)$, $A' = \text{Spec}_{\alpha/2}(H)$. Write $L = \frac{|A'|}{|A|}$.

③ Ben Green 4/23 Then, for any $h \in H$, $\omega(A, h \cdot A) \geq \frac{\alpha^4}{L}$.

Pf. For each $x \in A'$ write $r(x)$ for the # pairs $a_1, a_2 \in A$ with $a_1 - a_2 = x$.

Showed $\sum_{x \in A'} r(x) \geq \frac{\alpha^2}{2} |A|^2$. Applying Cauchy gives

$$\sum_{x \in A'} r(x)^2 \geq \frac{\frac{\alpha^4}{4} |A|^4}{|A|} \Rightarrow \# (\text{solutions to } a_1 + a_2 = a_3 + a_4) \geq \frac{\alpha^4}{4} |A|^3$$

$\Rightarrow \omega(A, A) \geq \frac{\alpha^4}{4L}$. To get the claim, note that A is H -invariant. ■

Combining this with multiplicative BSG gives:

Corollary: Let $H \leq \mathbb{F}_p^*$, $0 \leq \alpha \leq 1$, $A = \text{Spec}_\alpha(H)$, $A' = \text{Spec}_{\alpha/2}(H)$.

$L = \frac{|A'|}{|A|}$. Using approximate notation \approx at scale L , I can

find $x \in A$, $|x| \approx |A|$ and a set $H' \subseteq H$, $|H'| \approx |H|$, such that

$\delta[x, hx] \approx 1$ for all $h \in H'$.


Proof: "Dyadic chaining argument". Suppose $|H| = p^d$ and, as a hypothesis for contradiction, that $\text{Spec}_\eta(H) = \{0\}$ for some $\eta = p^{-o(1)}$.

Let J be a quantity (integer) to be specified later.

Set $d_0 := \eta$, $d_1 = \frac{d_0^2}{2}$... , $d_{j+1} = \frac{d_j^2}{2}$, up to d_j

Then $\text{Spec}_{d_0}(H) \subseteq \dots \subseteq \text{Spec}_{d_j}(H)$, By pigeonhole there is

some j such that $\frac{|A|}{|A'|} \leq p^{1/J}$ where $A = \text{Spec}_{d_j}(H)$, $A' = \text{Spec}_{d_{j+1}}(H)$.

Work out details: by taking $J = J(\delta)$ sufficiently large, get contradiction unless. 

Burgain-Katz-Tao (Sketch):

Suppose $A \subseteq \mathbb{F}_p$, $p^\delta \leq |A| \leq p^{1-\delta}$. Suppose $|A \cdot A| \approx |A+A| \approx |A|$ where the rough notation parameter $\approx = p^{o(1)}$. By the lemma about sets S_i that I proved earlier, there is an a_0 and a set $A' \subseteq A$,

$|A'| \approx |A|$ such that $|a \cdot A \cap a_0 A| \approx |A|$ for all $a \in A'$.

$$\Rightarrow |A \cap \frac{a_0}{a} A| \approx |A|.$$

From Ruzsa calculus we have $\delta(A, \frac{a_0}{a} A) \approx 1$, contradicting

the sum product estimate " $|A+bA| \geq |A|^{1+c\delta}$ ".