Bourgain–Glibichuk–

$H \leq \mathbb{F}_p^*$, $|H| \geq p^\delta$ then $\sup_{\xi \neq 0} \frac{1}{|H|} \left| \sum_{x \in H} e\left(\frac{x\xi}{p}\right) \right| \ll p^{-\delta'}$, $\delta' = \delta'(\delta) > 0$

Recall, if $H \subseteq \mathbb{F}_p$ is any set, write $\mathrm{Spec}_\alpha(H) := \{\xi : \left|\sum_{x \in H} e\left(\frac{x\xi}{p}\right)\right| \geq \alpha |H|\}$

Lemma: Suppose $B \subseteq \mathrm{Spec}_\alpha(H)$, then for a proportion at least $\frac{\alpha^2}{2}$ of the pairs $x, y \in B$ we have $x - y \in \mathrm{Spec}_{\alpha^2/2}(H)$.

Pf: Suppose $B = \{\xi_1, \ldots, \xi_k\}$. For each $j$ choose a complex # $c_j$ with $|c_j| = 1$, such that $c_j \sum_{x \in H} e\left(\frac{x\xi_j}{p}\right)$ is real and at least $\alpha |H|$.

Summing over $j$ and swapping summation order:

$$\sum_{x \in H} \sum_{j=1}^{k} c_j e\left(\frac{\xi_j x}{p}\right) \geq k\alpha |H|$$

, Applying C-S gives:

$$\sum_{x \in H} \sum_{1 \leq j, j' \leq k} c_j \overline{c_{j'}} e\left(\frac{(\xi_j - \xi_{j'})x}{p}\right) \geq k^2 \alpha^2 |H|. \quad \Rightarrow$$

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$$\Rightarrow \sum_{j,j'} \left| \sum_{x\in H} e\left(\frac{(\xi_j-\xi_{j'})x}{p}\right)\right| \geq k^2\alpha^2|H|.$$ It follows easily that

for at least $\frac{\alpha^2 k^2}{2}$ pairs $j,j'$ we have $\left|\sum_{x\in H} e\left(\frac{(\xi_j-\xi_{j'})x}{p}\right)\right| \geq \frac{\alpha^2|H|}{2}$. $\square$

## Additive-Multiplicative B-S-G Theorem

Suppose $A\subseteq \mathbb{F}_p$ and $B\subseteq \mathbb{F}_p^*$ such that $\omega(A, b\cdot A)\approx 1$ for all $b\in B$.

Then there are sets $A'\subseteq A$, $B'\subseteq xB$ (for some $x\neq 0$) such that

$\delta[A', bA']\approx 1$ for all $b\in B'$ and $|A'|\approx |A|$
$|B'|\approx|B|$.

Pf: By the usual B-S-G, there are, for each $b\in B$, subsets $X_b\subseteq A$ and $Y_b\subseteq A$ such that $|X_b|\approx|Y_b|\approx|A|$ and $\delta(X_b, b\cdot Y_b)\approx 1$.

The aim is to remove dependence on $b$.

<u>Lemma</u>: Suppose $S$ is a finite set and $S_1,\dots, S_k$ are subsets of $S$ with $|S_i|\geq\delta|S|$. Then there is an $i$ such that

$|S_i\cap S_j|\geq\frac{\delta^2}{2}|S|$ for at least $\frac{\delta^2 k}{2}$ values of $j$.

Proof: Cauchy-Schwarz: exercise. $\square$

Apply this lemma to sets $(X_b \times Y_b)_{b\in B}$ which are subsets of $A\times A$

of size $\approx |A\times A|$. So $\delta \approx 1$. I get a $b_0 \in B$ such that

$|X_{b_0}\cap X_b|, |Y_{b_0}\cap Y_b| \approx |A|$ for a set $B'$ of $b$'s with $|B'| \approx |B|$.

I also know that $X_b \sim b.Y_b$ and $X_{b_0} \sim b_0 \cdot Y_{b_0}$ (where recall $U\sim V \iff \delta[U,V] \approx 1$)

From Rusza Calculus we have: $\delta[X_b], \delta[X_{b_0}], \delta[Y_b], \delta[Y_{b_0}] \approx 1$.

Also, by statement (iii) in Rusza calculus, we have $X_{b_0}\sim X_b$ and $Y_{b_0}\sim Y_b$

$\Downarrow$

Also, $X_{b_0}\sim X_b \sim b\cdot Y_b \longrightarrow bY_{b_0}\sim bY_b$

Hence $bY_{b_0} \sim b_0 Y_{b_0} \Rightarrow Y_{b_0} \sim \frac{b}{b_0}\cdot Y_{b_0}$

Taking $A' = Y_{b_0}$ and redefining $B'$ to be this set of $\frac{b}{b_0}$'s, we

conclude the result. $\blacksquare$

Lemma: Let $H \leqslant \mathbb{F}_p^*$ be a multiplicative subgroup. Let $0 \leqslant \alpha \leqslant 1$,
let $A = \mathrm{Spec}_\alpha(H)$, $A' = \mathrm{Spec}_{\alpha^2/2}(H)$. Write $L = \frac{|A'|}{|A|}$.

Then, for any $h\in H$, $\omega(A, h\cdot A) \geq \frac{\alpha^4}{L}$.

Pf: For each $x \in A'$ write $r(x)$ for the # pairs $a_1, a_2 \in A$ with $a_1 - a_2 = x$.

Showed $\sum_{x \in A'} r(x) \geq \frac{\alpha^2}{2}|A|^2$. Applying Cauchy gives

$$\sum_{x \in A'} r(x)^2 \geq \frac{\frac{\alpha^4}{4}|A|^4}{|A'|} \Rightarrow \#\left(\text{solutions to } a_1 + a_2 = a_3 + a_4\right) \geq \frac{\alpha^4}{4L}|A|^3$$

$\Rightarrow \omega(A, A) \geq \frac{\alpha^4}{4L}$. To get the claim, note that $A$ is $H$-invariant. ∎

Combining this with multiplicative B-S-G gives:

Corollary: Let $H \leq \mathbb{F}_p^*$, $0 < \alpha < 1$, $A = \mathrm{Spec}_\alpha(H)$, $A' = \mathrm{Spec}_{\alpha^2/2}(H)$.

$L = \frac{|A'|}{|A|}$. Using approximate notation ~~of~~ at scale $L$, I can find $X \subseteq A$, $|X| \approx |A|$ and a set $H' \subseteq H$, $|H'| \approx |H|$, such that $\delta[X, hX] \approx 1$ for all $h \in H'$.

Proof: "Dyadic chaining argument". Suppose $|H| = p^\delta$ and, as a hypothesis for contradiction, that $\mathrm{Spec}_\eta(H) \neq \{0\}$ for some $\eta = p^{-o(1)}$?

Let $J$ be a quantity (integers) to be specified later.

Set $\alpha_0 := \eta$, $\alpha_1 = \frac{\alpha_0^2}{2}$ $\cdots$, $\alpha_{j+1} = \frac{\alpha_j^2}{2}$, up to $\alpha_J$

Then $\mathrm{Spec}_{\alpha_0}(H) \subseteq \cdots \subseteq \mathrm{Spec}_{\alpha_J}(H)$. By pigeonhole there is some $j$ such that $\frac{|A'|}{|A|} \leq p^{1/J}$ where $A = \mathrm{Spec}_{\alpha_j}(H)$, $A' = \mathrm{Spec}_{\alpha_{j+1}}(H)$.

Work out details: by taking $J = J(\delta)$ sufficiently large, ~~get~~ contradiction ensues. ∎

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<u>Bourgain-Katz-Tao</u> (Sketch):

Suppose $A \subseteq \mathbb{F}_p$, $p^\delta < |A| < p^{1-\delta}$. Suppose $|A \cdot A| \approx |A+A| \approx |A|$ where the rough notation parameter $K = p^{o(1)}$. By the lemma about sets $S_i$ that I proved earlier, there is an $a_0$ and a set $A' \subseteq A$, $|A'| \approx |A|$ such that $|a \cdot A \cap a_0 A| \approx |A|$ for all $a \in A'$.

$$\Rightarrow |A \cap \tfrac{a_0}{a} A| \approx |A|.$$

From Ruzsa calculus we have $\delta(A, \frac{a_0}{a} A) \approx 1$, contradicting the sum product estimate "$|A + bA| \geq |A|^{1+c\alpha\beta}$".

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