11/2 Ben Green

Balog–Szemerédi–Gowers:

$A, B \subseteq G$ have $\omega(A,B) \geq \frac{1}{K}$. Then there are $A' \subseteq A$, $B' \subseteq B$

with $\frac{|A'|}{|A|}, \frac{|B'|}{|B|} \geq K^{-C}$ such that $\sigma[A',B'] \leq K^{C}$.

Deduction of BSG:

Suppose A, B have $\omega(A,B) \geq \frac{1}{K}$. Easy to show that $K^{-2}|A| \leq |B| \leq K^2|A|$. (*)

(Note that $\omega(A,B) \leq \frac{|A|^2|B|}{|A|^{3/2}|B|^{3/2}}, \frac{|A|\cdot|B|^2}{|A|^{3/2}|B|^{3/2}}$.)

Define $r(x) = \#\{(a,b) \in A \times B;\ a-b=x\}$

Say "x is a popular difference" if $r(x) \geq \frac{1}{2K}|A|^{1/2}|B|^{1/2}$.

By an argument seen before (noting that $\omega(A,B) = \frac{\sum_x r(x)^2}{|A|^{3/2}|B|^{3/2}}$) one

sees there are at least $\frac{1}{2K}|A|^{1/2}|B|^{1/2}$ popular differences in $A-B$.

We'll create a graph as follows. Join $a \in A$ to $b \in B$ if $a-b$ is popular.

Add some extra vertices to the smaller class, and no extra edges, to

create a bipartite graph with two vertex classes of

size $n = \max(|A|, |B|)$. (cf. (*))

By preceding discussion, the edge density of this graph, α, is at least K^{-C}. By the lemma about paths of length 3, find $A' \subseteq A$, $B' \subseteq B$ with $\frac{|A'|}{|A|}, \frac{|B'|}{|B|} \geq K^{-C}$ such that the following is true:

$\forall a' \in A', b' \in B'$ there are $\geq K^{-C} n^2$ choices of $a'' \in A$ and $b'' \in B$ such that

$a'-b''$, $a''-b''$, and $a''-b'$ are all popular.

Observe that $a'-b' = (a'-b'') - (a''-b'') + (a''-b')$.

That is I can write $a'-b'$ as $x-y+z$ where x, y, z are all popular differences, in at least $K^{-C} n^2$ different ways.

(Exercise: they really are different ways.)

Noting that there are at most $\leq K^{C} n$ popular differences, we have $|A'-B'| \cdot K^{-C} n^2 \leq (K^{C} n)^3 \implies |A'-B'| \leq K^{C'} n$. $\square$

§4 Progressions of length 4 and the U^3-norm

Gowers norms: Let G be a finite abelian group and let $f : G \to \mathbb{C}$. Then we define the (Gowers) U^k-norm:

$$\|f\|_{U^k} := \left(\mathbb{E}_{x \in G, \vec{h} \in G^k} \prod_{\omega \in \{0,1\}^k} C^{|\omega|} f(x + \omega \cdot \vec{h}) \right)^{1/2^k}$$

where $\omega \cdot \vec{h} = \omega_1 h_1 + \dots + \omega_k h_k$, $|\omega| = \omega_1 + \dots + \omega_k$

and C is the complex conjugation operator.

Also introduce so-called Gowers inner product. If $(f_\omega)_{\omega \in \{0,1\}^k}$ is a collection of 2^k functions, define

$$\langle (f_\omega)_{\omega \in \{0,1\}^k} \rangle_{U^k} := \mathbb{E}_{x, \vec{h}} \prod_{\omega \in \{0,1\}^k} C^{|\omega|} f_\omega(x + \omega \cdot \vec{h}),$$

e.g. $$\langle f_{00}, f_{01}, f_{10}, f_{11} \rangle_{U^2} = \mathbb{E}_{x, h_1, h_2} f_{00}(x) \overline{f_{10}(x+h_1) f_{01}(x+h_2)} f_{11}(x+h_1+h_2)$$

__Lemma__ (Basic property of Gowers norm and inner product).

(i) (Gowers Cauchy–Schwarz ineq)

$$\left|\langle (f_\omega)_{\omega\in\{0,1\}^k}\rangle_{U^k}\right| \le \prod_\omega \|f_\omega\|_{U^k}$$

(ii) $\|f\|_{U^2} \le \|f\|_{U^3} \le \cdots$

(iii) The $\|\cdot\|_{U^k}$ norms are norms if $k\ge 2$.

That is $\|f\|_{U^k}=0 \iff f=0$ and $\|f+g\|_{U^k}\le \|f\|_{U^k}+\|g\|_{U^k}$.

"Proof" — We'll do special case of the U^2 norm. Higher norms are similar, but notationally worse.

Note that $\langle f_{00},f_{10},f_{01},f_{11}\rangle_{U^2} = \mathbb{E}_{x,x',y,y'\in G}\, f_{00}(x+y)\, f_{10}(x'+y)\, \overline{f_{01}(x+y')\, f_{11}(x'+y')}$

(reparameterization.)

Applying Cauchy-Schwarz shows that

$$|\text{this}| \leqslant \left(\mathbb{E}_{y,y'}\left(\mathbb{E}_x f_{00}(x+y)\overline{f_{01}(x+y')}\right)^2\right)^{1/2}\left(\mathbb{E}_{x'} \overline{f_{10}(x'+y)}\, f_{11}(x'+y')\right)$$

$$\leqslant \left(\mathbb{E}_{y,y'}\left|\mathbb{E}_{x} f_{00}(x+y)\overline{f_{01}(x+y')}\right|^2\right)^{1/2} \cdot$$

$$\cdot \left(\mathbb{E}_{y,y'}\left|\mathbb{E}_{x'} \overline{f_{10}(x'+y)}\, f_{11}(x'+y')\right|^2\right)^{1/2}$$

$$= \left|\langle f_{00}, f_{01}, f_{00}, f_{01}\rangle_{U^2}\right|^{1/2} \cdot \left|\langle f_{10}, f_{11}, f_{10}, f\rangle_{U^2}\right|^{1/2}$$

By a similar argument $\left|\langle f_{00}, f_{01}, f_{00}, f_{01}\rangle_{U^2}\right| \leqslant \left|\langle f_{00}, f_{00}, f_{00}, f_{00}\rangle_{U^2}\right|^{1/2} \cdot$

$$\cdot \left|\langle f_{01}, f_{01}, f_{01}, f_{01}\rangle_{U^2}\right|^{1/2}$$

$$= \|f_{00}\|_{U^2}^2 \cdot \|f_{01}\|_{U^2}^2$$

also $\left|\langle f_{10}, f_{11}, f_{10}, f_{11}\rangle_{U^2}\right| \leqslant \|f_{10}\|_{U^2}^2 \cdot \|f_{11}\|_{U^2}^2$

(5) Putting this together gives (i).

(ii) To see, for example, ~~that~~ that $\|f\|_{U^2} \leq \|f\|_{U^3}$

note that $\langle f,f,f,f,1,1,1,1\rangle_{U^3} = \|f\|_{U^4}^4$

Hence, by (i), $\|f\|_{U^4}^4 = \langle f \ldots 1\rangle_{U^3} \leq \|f\|_{U^3}^4 \|1\|_{U^3}^4 = \|f\|_{U^3}^4$.

(iii) The ~~fact that~~ $\|f\|_{U^k} = 0 \iff f = 0$ now follows from ~~that assertion~~ in the case $k=2$. One way to see this is to use the fact $\|f\|_{U^2} = \|\hat{f}\|_4$. Thus $\|f\|_{U^2} = 0$

$\Rightarrow \hat{f} = 0 \Rightarrow f = 0$.

To get the Δ-ineq, start with $\|f+g\|_{U^k}^{2^k} = \langle f+g, \ldots, f+g\rangle_{U^k}$.

Using multi-linearity of Gowers inner product, we may expand as a sum of 2^{2^k} terms of the form $\langle f, g, g, f, \ldots\rangle_{U^k}$; these being $\binom{2^k}{i}$ terms with i copies of f. Applying (i) shows that such a term is $\leq \|f\|_{U^k}^i \cdot \|g\|_{U^k}^{2^k - i}$.

Thus $\|f+g\|_{U^k}^{2^k} \leqslant \sum_i \binom{2^k}{i} \|f\|_{U^k}^{i} \|g\|_{U^k}^{2^k-i}$

$$= \left(\|f\|_{U^k} + \|g\|_{U^k}\right)^{2^k}. \quad \square$$