Local inverse thm for U^3

Suppose $f: \mathbb{Z}/N'\mathbb{Z} \to \mathbb{C}$, 1-bounded, $\|f\|_{U^3} \geq \delta$.

Then $\exists$ A.P. $Q \subseteq \mathbb{Z}/N'\mathbb{Z}$, $|Q| \geq \exp\left(-\frac{1}{\delta^c}\right) N^{\delta^c}$, and quadratic phases $\psi_i : \mathbb{Z}/N'\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $i = 0, \dots, N'-1$

such that $\mathbb{E}_i \left| \mathbb{E}_{x \in Q+i} f(x) e(-\psi_i(x)) \right| \gg \delta^c$ ... $(*)$

Putting this together with what we did on 4APs, get:

Prop: Suppose $A \subseteq [N]$, $|A| = \alpha N$, has no non-trivial 4APs. $N \geq \alpha^{-c}$. Write $f = 1_A - \alpha 1_{[N]}$ for the balanced function f; then f satisfies $(*)$.

Need to proceed from $(*)$ to something like $\mathbb{E}_{x \in P} f(x) \gg \alpha^c$ for some P.

this gives the density increment prop.

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§ A little Diophantine approximation.

Thm Let $\theta \in \mathbb{R}/\mathbb{Z}$. Let $N \geq 2$. Then there is $n \leq N$ such that $\|n^2\theta\|_{\mathbb{R}/\mathbb{Z}} \leq N^{-c}$.

Remark: With $\|n\theta\|_{\mathbb{R}/\mathbb{Z}}$ instead of $\|n^2\theta\|_{\mathbb{R}/\mathbb{Z}}$, this follows from the usual application of the pigeonhole principle with $c = 1$.

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Alternative formulation I: Every θ can be well-approximated by rationals with square denominator.

Alternative II: Squares are a "Poincaré sequence".
Van der Corput.

Remark: I can get $\|n^2\theta\|_{\mathbb{R}/\mathbb{Z}} \leq \omega(N) \to 0$ by applying Roth's theorem.

Take $\varepsilon > 0$, and split $\mathbb{R}/\mathbb{Z}$ into $\lceil 1/\varepsilon \rceil$ intervals of length $\leq \varepsilon$. Call them $I_1, \dots, I_m$. Define $A_j \subseteq [N]$ by $A_j = \{n \in [N] : n^2\theta \in I_j \pmod 1\}$.

One of these A_j must contain a 3AP (by Roth). Thus

$$n^2\theta,\ (n+d)^2\theta,\ (n+2d)^2\theta \in I_j \pmod 1.$$

Then look at $n^2\theta - 2(n+d)^2\theta + (n+2d)^2\theta = 2d^2\theta$;

we have $\|2d^2\theta\|_{\mathbb{R}/\mathbb{Z}} \leq 4\varepsilon$.

Key ingredient:

Proposition (Weyl's inequality): Let $\alpha, \beta, \gamma \in \mathbb{R}/\mathbb{Z}$ and suppose $\left|\mathbb{E}_{n \leq N}\, e(\alpha n^2 + \beta n + \gamma)\right| \geq \delta$, then α must be "diophantine", specifically $\exists q$, $1 \leq q \leq \delta^{-c}$, such that $\|\alpha q\|_{\mathbb{R}/\mathbb{Z}} \leq \delta^{-c}/N^2$.

Pf (of Weyl): Squaring, making a substitution and Δ-inequality gives

$$\mathbb{E}_{|h|\leq N}\left|\mathbb{E}_{n\in I_h}\, e\left(\alpha(n+h)^2-\alpha n^2\right)\right| \gg \delta^2$$

for intervals $I_h \subseteq [N]$.

$$\Rightarrow \mathbb{E}_{|h|\leq N}\left|\mathbb{E}_{n\in I_h}\, e(2\alpha h n)\right| \gg \delta^2.$$

The inner sum is simply a geometric progression (G.P.) with sum

$$\leq \frac{2}{\|1-e(2\alpha h)\|} \ll \|2\alpha h\|^{-1}_{\mathbb{R}/\mathbb{Z}} \quad (\text{and, trivially, } \leq N).$$

It follows that, for some C_0, there are $\gg \delta^{C_0}N$ values of $h \in [N]$ such that $\|\alpha h\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{\delta^{-C_0}}{N}$.

Let C_1 be an absolute constant to be specified. ~~[Apply pigeonhole principle, there is a q, $1\leq q\leq \delta^{C_1}N^2$ such that]~~

Let $Q=\delta^{C_1}N^2$. By pigeonhole principle, there is q, $1\leq q\leq Q$, such that $\|\alpha q\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{Q}$. If $\delta \leq \delta^{-C_1}$ then we are done.

Suppose this is not the case, ie $\delta^{-C_1} \leq q \leq \delta^{C_1}N^2$.

There is an a such that $\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{qQ}$.

Let h, h' be two elements in some interval of length $\frac{q}{2}$.

Then $\|\alpha h - \alpha h'\|_{\mathbb{R}/\mathbb{Z}} \geq \left|\frac{a(h-h')}{q}\right| - \frac{1}{2Q} \geq \frac{1}{q} - \frac{1}{2Q} \geq \frac{1}{2q}$.

(note that $ah \neq ah' \pmod q$) since we may assume wlog that $\text{h.c.f}(a,q) = 1$.)

Thus, as h ranges over any interval of length $\frac{q}{2}$, the fractional parts $\|\alpha h\|_{\mathbb{R}/\mathbb{Z}}$ are $\frac{1}{2q}$-separated. The number with $\|\alpha h\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{\delta^{-c_0}}{N}$ is therefore $O\left(1 + \frac{q\delta^{-c_0}}{N}\right)$. (think about it)

Furthermore, $[N]$ may be divided into $O\left(1+\frac{N}{q}\right)$ intervals of length $\leq \frac{q}{2}$. It follows that the number of $h \in [N]$ with $\|\alpha h\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{\delta^{-c_0}}{N}$ is bounded above by $O\left(1+\frac{N}{q}\right)\left(1+\frac{q\delta^{-c_0}}{N}\right) =$

$$= O\left(1 + \frac{N}{q} + \frac{q\delta^{-c_0}}{N} + \delta^{-c_0}\right).$$

But recall that $q \geq \delta^{-c_1}$, $q \leq \delta^{c_1} N^2$ and (I forgot to say) $N > \delta^{-c}$. This is for an appropriate choice of c_1, much less than $\delta^{c_0} N$ # ← (contradiction).

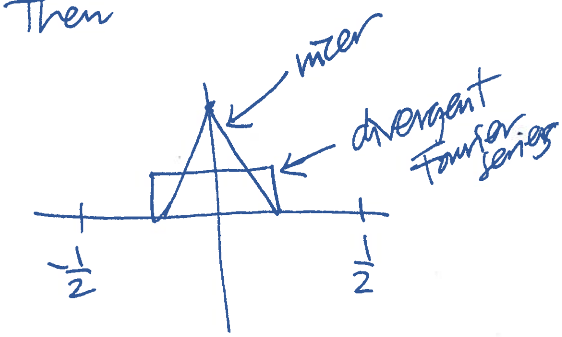
$\square$

Proof of Theorem: Suppose false, then for arbitrary small ε, we have $\|n^2\theta\|_{\mathbb{R}/\mathbb{Z}} > \varepsilon$ for all $n \leq \varepsilon^{-C_0}$.

Take a function $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}_{\geq 0}$ with following properties:

- $\|\psi\|_1 = 1$ (mean)
- $\psi(x) = 0$ for $\|x\|_{\mathbb{R}/\mathbb{Z}} \geq \varepsilon$
- If $\widehat{\psi}(r) = \int_0^1 \psi(x) e^{-2\pi i x r}\, dx$, then

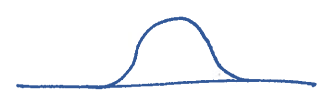
$$\sum_{|r| \geq \varepsilon^{-C}} |\widehat{\psi}(r)| \leq \frac{\varepsilon}{2}$$



(e.g. Fejér "tent function") Fejér

$e^{-\frac{1}{1-x^2}}$

bump function in diff. geo.



$\psi = \psi_1 * \psi_2 * \psi_3 * \ldots$ infinite

hard to compute Fourier series or upper bound on them.

$\psi_i = 1_{[-\frac{1}{i^2}, \frac{1}{i^2}]}$