

## Local inverse thm for $U^3$

Suppose  $f: \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ , 1-bounded,  $\|f\|_{U^3} \geq \delta$ .

Then  $\exists$  A.P.  $Q \subseteq \mathbb{Z}/N\mathbb{Z}$ ,  $|Q| \geq \exp(-\frac{1}{\delta^c}) N^{\delta^c}$ , and quadratic phases  $\psi_i: \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ ,  $i=0, \dots, N^{\delta^c}-1$

such that  $\mathbb{E}_i \left| \mathbb{E}_{x \in Q} f(x) e(-\psi_i(x)) \right| \gg \delta^c \dots (*)$

Putting this together with what we did on 4APs, get:

Prop: Suppose  $A \subseteq [N]$ ,  $|A| = \alpha N$ , has <sup>no</sup> non-trivial 4APs.

$N \geq \alpha^{-c}$ . Write  $f = 1_A - \alpha 1_{[N]}$  for the balanced function  $f$ ; then  $f$  satisfies  $(*)$ .

Need to proceed from  $(*)$  to something like  $\mathbb{E}_{x \in P} f(x) \gg \alpha^c$  for some  $P$ .  
this gives the density increment prop.

## § A little Diophantine approxn.

Thm Let  $\theta \in \mathbb{R}/\mathbb{Z}$ . Let  $N \geq 2$ . Then there is  $n \leq N$

such that  $\|n^2 \theta\|_{\mathbb{R}/\mathbb{Z}} \leq N^{-c}$ .

Remark: With  $\|n\theta\|_{\mathbb{R}/\mathbb{Z}}$  instead of  $\|n^2\theta\|_{\mathbb{R}/\mathbb{Z}}$ , this follows from the usual approximation of the pigeonhole principle with  $c=1$ .

Alternative formulation I: Every  $\theta$  can be well-approximated by rationals with square denominator.

Alternative II: Squares are a "Poincaré sequence".  
Vander Corput.

Remark: I can get  $\|n^2\theta\|_{\mathbb{R}/\mathbb{Z}} \leq \omega(N) \rightarrow 0$  by applying Roth's theorem.

Take  $\varepsilon > 0$ , and split  $\mathbb{R}/\mathbb{Z}$  into  $\lceil 1/\varepsilon \rceil$  intervals of length  $\leq \varepsilon$ . Call them  $I_1, \dots, I_m$ . Define  $A_j \subseteq [N]$  by  $A_j = \{n \in [N] : n^2\theta \in I_j \pmod{1}\}$ .

One of these  $A_j$  must contain a 3AP (by Roth). Thus

$$n^2\theta, (n+d)^2\theta, (n+2d)^2\theta \in I_j \pmod{1}.$$

$$\text{Then look at } n^2\theta - 2(n+d)^2\theta + (n+2d)^2\theta = 2d^2\theta;$$

$$\text{we have } \|2d^2\theta\|_{\mathbb{R}/\mathbb{Z}} \leq 4\varepsilon.$$

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Key ingredient:

Proposition (Weyl's inequality): Let  $\alpha, \beta, \gamma \in \mathbb{R}/\mathbb{Z}$  and suppose

$$\left| \sum_{n \leq N} e(\alpha n^2 + \beta n + \gamma) \right| \geq \delta, \text{ then } \alpha \text{ must be "disproportionate"}$$

$$\text{specifically } \exists q, 1 \leq q \leq \delta^{-c}, \text{ such that } \| \alpha q \|_{\mathbb{R}/\mathbb{Z}} \leq \frac{\delta^{-c}}{N^2}.$$

R (of Weyl): Squaring, making a substitution and  $\Delta$ -inequality gives

$$\mathbb{E}_{|h| \leq N} \left| \mathbb{E}_{n \in I_h} e(\alpha(n+h)^2 - \alpha n^2) \right| \geq \delta^2$$

for intervals  $I_h \subseteq [N]$ .

$$\Rightarrow \mathbb{E}_{|h| \leq N} \left| \mathbb{E}_{n \in I_h} e(2\alpha h n) \right| \geq \delta^2.$$

The inner sum is simply a geometric progression (G.P.) with sum

$$\leq \frac{2}{|1 - e(2\alpha h)|} \ll \|2\alpha h\|_{\mathbb{R}/\mathbb{Z}}^{-1} \quad (\text{and, trivially, } \ll N).$$

It follows that, for some  $C_0$ , there are  $\gg \delta^{C_0} N$  values of  $h \in [N]$

$$\text{such that } \|2\alpha h\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{\delta^{C_0}}{N}.$$

Let  $C_1$  be an absolute constant to be specified. ~~[Apply pigeonhole principle,~~

~~there is a  $q, 1 \leq q \leq \delta^{C_1} N^2$  such that]~~

Let  $Q = \delta^{C_1} N^2$ . By pigeonhole principle, there is  $q, 1 \leq q \leq Q$  such that  $\|\alpha q\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{Q}$ . If  $\delta \leq \delta^{-C_1}$  then we are done.

Suppose this is not the case, i.e.  $\delta^{-C_1} \leq q \leq \delta^{C_1} N^2$ .

There is an  $a$  such that  $|\alpha - \frac{a}{q}| \leq \frac{1}{qQ}$ .

Let  $h, h'$  be two elements in some interval of length  $\frac{q}{2}$ .  
 Then  $\|ah - ah'\|_{\mathbb{R}/\mathbb{Z}} \geq \left| \frac{a(h-h')}{q} \right| - \frac{1}{2q} \geq \frac{1}{q} - \frac{1}{2q} \geq \frac{1}{2q}$ .

(note that  $ah \neq ah' \pmod{q}$ ) since we may assume wlog that  $\text{h.c.f.}(a, q) = 1$ .

Thus, as  $h$  ranges over any interval of length  $\frac{q}{2}$ , the fractional parts  $\|ah\|_{\mathbb{R}/\mathbb{Z}}$  are  $\frac{1}{2q}$ -separated. The number with

$\|ah\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{\delta^{-c_0}}{N}$  is therefore  $O\left(1 + \frac{q\delta^{-c_0}}{N}\right)$ . (think about it)

Furthermore,  $[N]$  may be divided into  $O\left(1 + \frac{N}{q}\right)$  intervals of length  $\leq \frac{q}{2}$ . It follows that the number of  $h \in [N]$  with

$\|ah\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{\delta^{-c_0}}{N}$  is bounded above by  $O\left(1 + \frac{N}{q}\right)\left(1 + \frac{q\delta^{-c_0}}{N}\right) =$

$= O\left(1 + \frac{N}{q} + \frac{q\delta^{-c_0}}{N} + \delta^{-c_0}\right)$ . But recall that  $q \geq \delta^{-c_1}$   
 $q \leq \delta^{c_1} N^2$

and (I forgot to say)  $N > \delta^{-c}$ . This is for an appropriate choice of  $c_1$ , much less than  $\delta^{c_0} N \neq \leftarrow$  (contradiction).  $\square$

Proof of Theorem: Suppose false, then for arbitrary small  $\varepsilon$ ,

we have  $\|h^{2n}\|_{\mathbb{R}/\mathbb{Z}} > \varepsilon$  for all  $n \leq \varepsilon^{-C}$ .

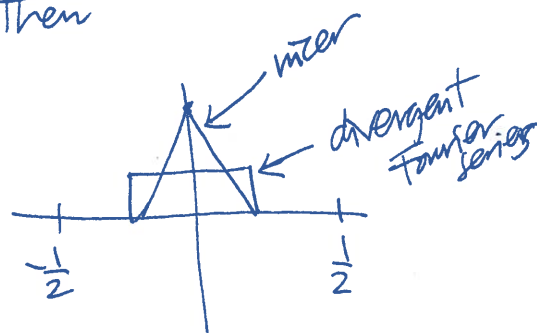
Take a function  $\psi: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}_{\geq 0}$  with following properties:

- $\|\psi\|_1 = 1$  (mean)

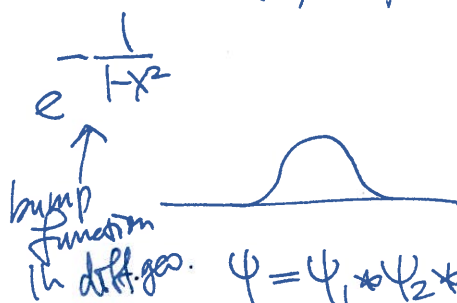
- $\psi(x) = 0$  for  $\|x\|_{\mathbb{R}/\mathbb{Z}} \geq \varepsilon$

- If  $\hat{\psi}(r) = \int_0^1 \psi(x) e^{-2\pi i r x} dx$ , then

$$\sum_{|r| \geq \varepsilon^{-C}} |\hat{\psi}(r)| \leq \frac{\varepsilon}{2}$$



(e.g. Fejér "tent function")



hard to compute Fourier series or upper bounds on them.

$\psi = \psi_1 * \psi_2 * \psi_3 * \dots$  infinite

$\psi_i = 1_{[-\frac{1}{2^i}, \frac{1}{2^i}]}$