

Ban Green 1/26

No Friday Lecture

Additive energy,

A, B sets in same abelian group.

Then $\omega_+(A, B) = \#\{(a_1 b_1, a_2 b_2) : a_1 + b_1 = a_2 + b_2, a_i \in A, b_i \in B\}$

$$\frac{|A|^{\frac{3}{2}} |B|^{\frac{3}{2}}}{|A|^{\frac{1}{2}} |B|^{\frac{1}{2}}}$$

Note: Possible to have $\omega_+(A, B) \geq \frac{1}{K}$ (say) whilst $\delta[A, B]$ is huge.

$$\frac{|A+B|}{|A|^{\frac{1}{2}} |B|^{\frac{1}{2}}}.$$

$$\text{Eg. } A=B=\{1, \dots, n\} \cup \{2^n, 2^{n+1}, \dots, 2^{2n-1}\}$$

(Ruzsa-Szemerédi) Gowers Thm:

Suppose $\omega_+(A, B) \geq \frac{1}{K}$. Then $\exists A' \subseteq A, B' \subseteq B$

with $\frac{|A'|}{|A|}, \frac{|B'|}{|B|} \geq K^{-C}$, such that $\delta[A', B'] \leq K^C$.

Rough notation with parameter K , $\omega_+(A, B) \approx 1 \Rightarrow \exists A', B'$ with
 $|A'| \approx |A|, |B'| \approx B$
and $\delta[A', B'] \approx 1$.

Remark: If $A=B$, then we can take $A'=B'$,

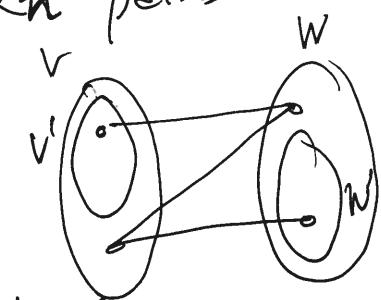
to see this apply B-S-G to get $A', A'' \subseteq A$

with $\delta[A', A''] \approx 1$. Ruzsa calculus $\Rightarrow \delta[A], \delta[A''] \approx 1$.

Prof.: An "application of graph theory".

Proposition: Let G be a bipartite graph on vertex sets $V \cup W$ (all edges b/w V and W) with $|V|=|W|$ and $c n^2$ edges. Then \exists sets $V' \subseteq V$ and $W' \subseteq W$ such that $\frac{|V'|}{|V|}, \frac{|W'|}{|W|} \geq \alpha^c$ and such that for every

$x \in V'$ and $y \in W'$ there are at least $c \alpha^{n^2}$ paths of length 3 between x and y in G .

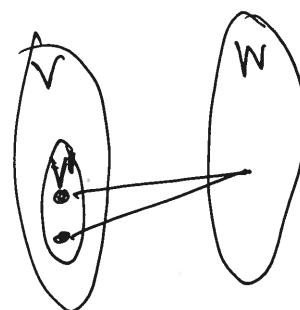


Deduce this from following lemma about paths of length 2.

Lemma: $G = V \cup W$ as before. Let $0 < \eta \leq 1$.

Then there is a set $V' \subseteq V$, $|V'| \geq \frac{\alpha n}{2}$. Such that for at least a proportion $1-\eta$ of the pairs $(v_1, v_2) \in V'$ there are at least $\frac{\eta \alpha^2}{2} n$ paths of length 2 between v_1 and v_2 .

Notation(?): If at least a proportion $1-\eta^2 =: \tilde{\eta}$



Proof: Recall that if $v \in V$ then by $N(v) = \text{neighborhood of } v$
 $= \text{edges in } W \text{ adjacent to } v.$

$E(G) = \text{set of edges of } G$. We have

$$\sum_{v \in V} \sum_{w \in N(v)} \frac{1}{\sum_{v' \in E(G)}} \geq \alpha \quad (*).$$

By Cauchy-Schwarz we thus have $\sum_{v \in V} \sum_{w \in N(v)} \frac{1}{\sum_{v' \in E(G)}} \geq \frac{\alpha^2 n}{\sum_{v \in V} |N(v)|}$

$$\Rightarrow \sum_{v, v' \in V} (N(v) \cap N(v')) \geq \alpha^2 n^2.$$

N.B.: $|N(v) \cap N(v')| = \# \text{paths of length 2 between } v \text{ and } v'$

Let $S \subseteq V \times V$ be set of "anti social" pairs, that is
 to say pairs with $|N(v) \cap N(v')| < \frac{\alpha^2 n^2}{2}$

Returning to (*), see that $\sum_{v \in V} (\eta - 1_{(v, v') \in S}) \sum_{w \in N(v)} \frac{1}{\sum_{v' \in E(G)}} \geq \frac{\alpha^2 n^2}{2}$

In particular, there is at least one $w \in W$ such that

$$\sum_{v \in V} (\eta - 1_{(v, v') \in S}) \sum_{w \in N(v)} \frac{1}{\sum_{v' \in E(G)}} \geq \frac{\alpha^2 n^2}{2}.$$

Take $v' = N(w)$. Then (ignoring the term $1_{(v, v') \in S}$ completely)

$$\text{we have } \sum_{v \in V} \sum_{w \in N(v)} \frac{1}{\sum_{v' \in E(G)}} \geq \frac{\alpha^2 n^2}{2}$$

$$\text{which implies } |N'| = |N(w)| \geq \left(\frac{\alpha^2}{2}\right)^{1/2} n \geq \frac{\alpha n}{2}.$$

Also, (simply using the fact that the RHS is ≥ 0)

we see that the proportion of antisocial pairs in V' is at most η . \square

Remark "Morely" We've chosen V' to be the neighborhood of a random vertex $w \in W$. This seems (in retrospect) sensible since we want V' to have good connectivity.

"Dependent random selection" — idea of Gowers.

Proof of proposition: $G = V \cup W$ with dn^2 edges. Delete all edges emanating from $v \in V$ with degree $\leq \frac{dn}{2}$. We still have $\frac{dn^2}{2}$ edges. Let G be this new graph. Apply the lemma on "paths of length 2" with η to be specified later, getting $V' \subseteq V$, $|V'| \geq \frac{dn}{4}$ such that if $(v, v') \in V$ we have at least $\frac{nd^3}{8}n$ vertices $w \in W$ with $v, v'w \in E(G)$. By passing to a further subset $V'' \subseteq V'$ if necessary with $|V''| \geq \frac{1}{2}|V'|$ (or inspecting proof of lemma) we can assume that all vertices of V'' have degree $\geq \frac{dn}{2}$.

There are $\geq \frac{dn}{8} \frac{dn}{2} = \frac{d^2 n^2}{16}$ edges from V'' to W .

By simple averaging argument, there is a set $W \subseteq V$, $|W| \geq \frac{\alpha^2}{32} n$

and for every $w \in W'$ there are at least $\frac{\alpha^2 n}{32}$ edges into V'' .

Looking at V'' , recalling that a proportion $\geq 1 - \eta$ of the pairs

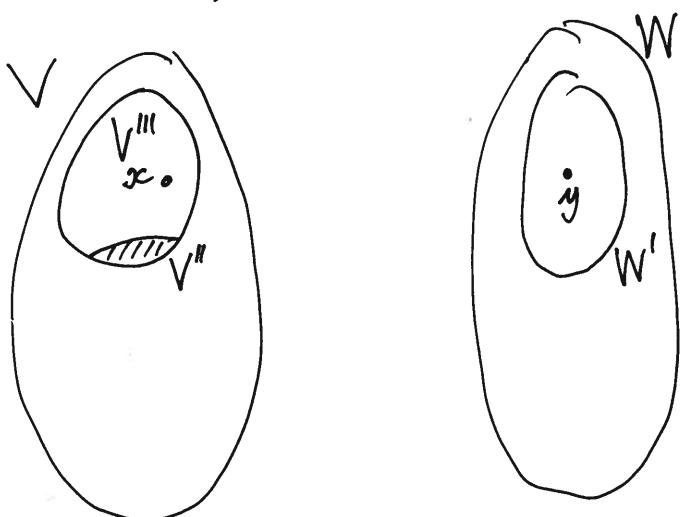
$v_1, v_2 \in V''$ are "sociable" in the sense that they have $\geq \frac{\eta \alpha^2}{8} n$

Common neighbors in G , pass to a further set $V''' \subseteq V''$,

$|V'''| \geq (1 - 2\eta) |V''|$ such that for every $x \in V'''$ there are

at least $(1 - 2\eta) |V''|$ vertices y in V'' such that x and y are
sociable.

Claim: There are many paths of length 3 (in G) between V''' and W !



Suppose $x \in V''$ and $y \in W'$.

$$|V'' \cap N(y)| \geq \frac{d_n}{32}$$

There are many $z \in V''$ such that x and z have many common neighbors t .

By choosing η small enough, $\eta < \frac{\alpha^2}{96}$ (something)

I can ensure significant overlap between the set of z 's and neighbors of y in V' .

Gives many paths $x \rightarrow t \rightarrow z \rightarrow y$ between x and y .

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