10/23 Ben Green

Chang's Covering Lemma:

Suppose $A \subseteq \mathbb{Z}$, $\delta[A] \ll K$, and $2A - 2A$ contains a proper G.A.P. P with dimension at most d and size $\geq \eta^{|A|}$.

Then A is contained in a GAP $\tilde{P}$ of dimension at most $d + CK^C \log\left(\frac{1}{\eta}\right)$ and size at most $2^d \eta^{-CK^C} |A|$.

Pf: Let $R_0 \subseteq A$ be maximal such that the translates $P + x$, $x \in R_0$ are all disjoint. Let L be a quantity to be specified later: $\left(L = K^{O(1)}.\right)$ If $|R_0| \leq L$ then stop. If not, then select $S_1 \subseteq R_0$ with $|S_1| = L$ and define $P_1 := P + S_1$.

Now let $R_1 \subseteq A$ be maximal such that the translates $P_1 + x$, $x \in R_1$ are all disjoint. If $|R_1| \leq L$ then stop. If not, select $S_2 \subseteq R_1$ with $|S_2| = L$ and define $P_2 := P_1 + S_2$...

Continue in this fashion.

Claim: if L chosen suitably then this stops in time $t \leq C \log \frac{1}{\eta}$.

Proof: Suppose not. Then we get $P_t = P + S_1 + \cdots + S_t$.

By construction, $|P_t| = L^t |P| \geq L^t \eta |A|$.

On the other hand, since $P \subseteq 2A - 2A$ and $S_i \subseteq A$, we have $|P_t| \leq |(t+2)A - 2A| \leq K^{ct} |A|$. (sumset inequalities)

Taking $L = K^{c'}$ with $c' \gg c$, this is a contradiction unless $t \leq c \log(\frac{1}{\eta})$.

What happens when algorithms terminates?

Then we have a set $R_t \subseteq A$ which is maximal s.t. the translates $P_t + x$, $x \in R_t$, are disjoint. and furthermore $|R_t| \leq L$.

By maximality, if $a \in A$ then $(P_t + a) \cap (P_t + x) \neq \emptyset$ for some $x \in R_t$ which implies that $A \subseteq P_t - P_t + R_t \subseteq$

$$\subseteq P - P + (S_1 - S_1) + \cdots + (S_t - S_t) + R_t.$$

We put the latter object inside a GAP.

$P-P$ is already a GAP of dimension d.

$S_i - S_i$ can be placed inside a GAP $\overline{S_i}$ with generators the elements of S_i and sidelength 3.

R_t can be put inside a GAP $\overline{R_t}$ with generators R_t and sidelength 2. Take $\tilde{P} := (P-P) + \overline{S_1} + \dots + \overline{S_t} + \overline{R_t}$.

Then $A \subseteq \tilde{P}$.

$$\dim(\tilde{P}) \leqslant d + 4(t+1) \leqslant d + CK^c \log\left(\frac{1}{\eta}\right)$$

and $|\tilde{P}| \leqslant |P-P|\,|\overline{S_1}| \cdots |\overline{S_t}|\,|R_t|$

$$\leqslant 2^d |P| 3^{4t} 2^4$$

$$\leqslant 2^d \eta^{-CK^c} |A| \text{ as required.} \quad \square$$

$\tilde{P}$ can be made proper with extra work.

Proof of Freiman Ruzsa: Suppose $A \subseteq \mathbb{Z}$, $\sigma[A] < K$.

(i) Apply Ruzsa model lemma to locate $A' \subseteq A$, $|A'| \geq \frac{1}{8}|A|$ which is Freiman 8-isomorphic to a set $S \subseteq \mathbb{Z}/p\mathbb{Z}$ where p prime, $p \leqslant CK^c|A|$

(ii) Apply Bogolyubov's lemma to locate inside $2S-2S$ a Bohr set $B(r_1 \dots r_k; \frac{1}{10})$ where $k \leq CK^C$

(iii) Apply geometry of numbers to find a proper GAP, P, inside that Bohr set of dimension $\leq CK^C$ and size $\geq \exp(-K^C)|A|$.

(iv) Lift this back to get a GAP of same dimension and size inside $2A'-2A' \subseteq 2A-2A$. [Used basic properties of Freiman homomorphisms.]

(v) Apply Chang's covering lemma to conclude. □

Remarks: $\sigma[A] \leq K \iff$ K-approximate subgroups of $\mathbb{Z}$.

Definition [Control]: Suppose A, B are sets in some abelian group. We say that A is K-controlled by B if $|B| \leq K|A|$ and if $A \subseteq B+X$ for some set X, $|X| \leq K$.

Rough classification problem:

Find a "small" / "algebraic" class of sets $\mathcal{B}$ which control all K-approximate groups.

Freiman-Ruzsa $\Leftrightarrow$ $\mathcal{B}$ consists of GAPs.

In fact, if $\sigma[A] \leq K$ then ~~an examination of the proof tells~~ A is $\exp(K^c)$-controlled by a GAP of dimension $\leq K^c$.

Conjecture (Polynomial Freiman-Ruzsa Conjecture):

A is K^c-controlled by a GAP$^{(*)}$ of dimension $\leq C \log K$.

(*) GAP = projection of lattice points in a box. Maybe use some other convex body. Geometry of numbers issue.

---

§4. Additive energy and Balog–Szemeredi–Gowers

Let A, B be two sets in some abelian group.

Then the normalized additive energy $\omega_+(A,B) :=$

$$\frac{\#\{a_1+b_1 = a_2+b_2 : a_1,a_2 \in A, b_1,b_2 \in B\}}{|A|^{3/2}\,|B|^{3/2}}.$$

Note that $0 \leqslant \omega_+(A,B) \leqslant 1$

To see the upper bound note that

$\#\{a_1+b_1 = a_2+b_2\} \leqslant |A|^2|B|$ since a_1, b_1, a_2 determines b_2 uniquely

$\leqslant |A+B|^2$.

[Compare to Young's inequality, a special case of which is

$\|f*g\|_2 \leqslant \|f\|_{1/3}\,\|g\|_{4/3}$ with $f = 1_A$ and $g = 1_B$.]

Abbreviate $\omega_+(A) = \omega_+(A,A)$.

~~Lemma~~

Define also $\delta[A,B] = \dfrac{|A+B|}{|A|^{1/2}|B|^{1/2}}$.

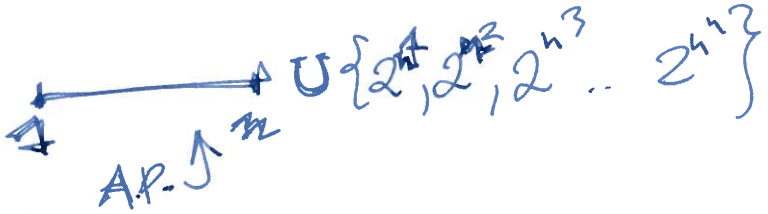
Lemma: Suppose $\delta[A,B] \leq K$, then $\omega_+(A,B) \geq \frac{1}{K}$.

Pf: Write $r(x) := \#\{(a,b): a+b=x\}$.

then $\sum_x r(x) = |A|\cdot|B|$, $\mathrm{supp}(r) = A+B$.

Cauchy–Schwarz $\Rightarrow \sum_x r(x)^2 \geq \dfrac{|A|^2|B|^2}{|A+B|}$.

This is equivalent to stated inequality. □



So there's no direct converse to lemma.

Thm (BSG): If $\omega_+[A,B] \geq \frac{1}{K}$ then $\exists A' \subseteq A, B' \subseteq B$,

$\dfrac{|A'|}{|A|}, \dfrac{|B'|}{|B|} \geq K^{-c}$ s.t. $\delta[A',B'] \leq K^c$.