

10/2 Ben Green

Rusza Calculus

$K \geq 2$ approximation parameter

$U, V \subseteq$ abelian group.

$U \sim V$ means

$$|U - V| \approx |U|^{1/2} |V|^{1/2}$$

$$|U - V| \leq c K^c |U|^k |V|^k$$

Rusza's third inequality:

Suppose $U, V, W \subseteq$ abelian group. And $U \sim V \sim W$.

Then $U + V \sim W$.

[This implies that if $\frac{|U - V|}{|U|^{1/2} |V|^{1/2}} \leq K$, and ditto for (V, W) ,

$$\text{then } |U + V + W| \leq \underbrace{f K^c}_{\text{since } k=2} |U|^{1/3} |V|^{1/3} |W|^{1/3}.$$

Corollary: Suppose $A \subseteq$ abelian group with $\sigma[A] \leq K$

(i.e. $\frac{|A+A|}{|A|} \leq K$). Then, for any integers $k, l \geq 0$,

$$\text{not both } 0, \quad |kA - lA| \leq \underbrace{f K^{\gamma(k,l)}}_{\text{very slow growth}} |A|$$

for some constant $\gamma(k, l)$.

Remark: Blaschke-Rusza inequalities allows us to take $\gamma(k, l) = k+l$.

Exercise: If A group (not abelian) we can have $\frac{|AA|}{|A|} \leq 3$

$$\text{but } \frac{|AAA|}{|A|} > 1000.$$

Prob 3rd Question:

S such that

The idea is to find any set $\check{S} \subseteq U + V$. Why does this suffice?

By Russa's calculus, it implies $\mathcal{C}[U + V] \approx 1$. Furthermore,

the fact that $U + V$ implies that there is some x such

that $|U \cap (x + V)| \geq |U|$. Indeed, writing $r(x)$ for $|U \cap (x + V)|$,

r is supported on $U - V$, which has cardinality $|U|/|V| (\approx |V|)$ and

and $\sum_x r(x) = |U| \cdot |V|$. Setting $x^* = x_0 + v$ for an

arbitrary $v \in V$, it follows that $|(U + V) \cap (x^* + V)| \geq |U|$.

Now apply part (iii)¹ of Russa Calculus, to deduce that $U + V \sim V$.

It remains to find S . We take S to be the set of "popular sums"

in $U + V$. Note that by R.C. we have $|U + V| \leq L|U|^{1/2} \cdot |V|^{1/2}$

where $L \leq K^3$. Take $S := \{x \in U + V \text{ such that } \tilde{r}(x) \geq \frac{1}{2L}|U|^{1/2}|V|^{1/2}\}$

where $\tilde{r}(x) := \#\{(u, v) : u + v = x\}$.

Claim: $|S| \geq \frac{1}{2L}|U|^{1/2}|V|^{1/2}$.

Proof: Use that $\sum_x f(x) = |U||V|$ and $f(x) \leq \min(|U|, |V|) \leq |W|^{\frac{1}{2}}|V|^{\frac{1}{2}}$ ■

Claim: $U+S+V$ is somewhat small.

Pf: Every element $u+s+v \in U+S+V$ can be written as at least $\frac{1}{2L} |U|^{\frac{1}{2}} |V|^{\frac{1}{2}}$ ways as $u + (u+v) + v$ (since $S \subseteq \text{pop. sum}$,
 $= (u+v) + (u+v)$)

this implies that $|U+S+V| \leq \frac{1}{2L} |U|^{\frac{1}{2}} |V|^{\frac{1}{2}} \leq |U+V|^{\frac{3}{2}}$.

hence $|U+S+V| \lesssim |U||V|$, etc. and therefore $S \subseteq U+V$. QED.

§3. Approximate groups and the Freiman-Ruzsa theorem.

Let K be an approximation parameter. By an "approximate group", I mean set $A \subseteq \text{abelian group}$ with $\delta(A) \leq K$.
 That is to say: $|A+A| \leq K|A|$.

What can be said about the structure
 of such sets?

Observe that if $A' \subseteq A$ and $|A'| \geq \delta|A|$, then

$\delta[A'] = \frac{|A'A'|}{|A'|} \leq \frac{|A+A|}{|A|} \leq \frac{K}{\delta}$. So the property "approximate group" is

somewhat hereditary, so rules out possibility for
 "rigid" structural classification.

Can we, however, find a nice class of objects & such that
 if $\mathcal{Z}[A] \leq K$ then A is economically contained in
 some $S \subseteq \mathcal{Z}$?

Theorem 1 (Ruzsa). Suppose $A \subseteq \mathbb{F}_2^\infty$ is a finite set with
 $\mathcal{Z}[A] \leq K$. Then A is a subspace of cardinality at
 most $\exp(K^c)|A|$.

Theorem 2 (Freiman-Ruzsa). Suppose $A \subseteq \mathbb{Z}$ is a
 finite subset of \mathbb{Z} with $\mathcal{Z}[A] \leq K$.

then $A \subseteq P$, a "Generalized arithmetic progression"
 with dimension at most K^c and size at most $\exp(K^c)|A|$
 By a G.A.P. of dimension d , we mean a set of the form:

$$P := \{x_0 + l_1 x_1 + \dots + l_d x_d : 0 \leq l_i < L_i\}$$

$L_1 \dots L_d$ are "sidelengths"

If $|P| = \text{size}(P)$, i.e. if all sums $x_0 + l_1 x_1 + \dots + l_d x_d$
 are distinct, then P is said to be proper.

Pf of Theorem I:

Suppose $A \subseteq \mathbb{F}_2^\infty$ has $\delta[A] \leq k$. Let $X \subseteq 3A = A + A + A$.

Such that the translates ~~$A+x$~~ $A+x$ are all disjoint
and such that X is maximal w.r.t. this property.

Note that $\bigcup_{x \in X} (A+x) \subseteq 4A$, a set of size $\leq k^c |A|$

by Ruzsa's 3rd inequality (or the corollary).

However the union is disjoint, whence $|X| \leq k^c$.

Suppose that $y \in 3A$. Then by the asserted maximality of X ,

$(A+x) \cap (A+y) \neq \emptyset$ for some $x \in X$.

This implies that $y \in A + A + x = 2A + x$.

Hence $3A \subseteq 2A + X$.

[Digression/exercise: $2A$ may not be covered by $O(k^{(1)})$
translates of A .]

Adding A to both sides gives $4A \subseteq 3A + X \subseteq 2A + 2X$ etc.

Continuing in this fashion, we see that

$\langle A, \text{the group spanned by } A \rangle$ is
contained in $2A + \langle X \rangle$.

Hence $|KA| \leq (2A) \cdot |\langle x \rangle|$
 $\leq K|A|2^{K^c}$. QED.

What are the best bounds in this theorem?

If $A \subseteq \mathbb{F}_2^\infty$, $\# \langle A \rangle \leq K$ then there may be no subspace H with $A \subseteq H$ and $|H| = 2^{\circ(K)} |A|$. Eg. $A = \{e_1 \dots e_d\}$.

Possibility: (Polynomial Freiman-Ruzsa Conjecture).

Under the same assumptions, there is H , $|H| \leq K^c |A|$ such that $|A \cap H| \geq K^{-c} |A|$.

Equivalent form: Suppose $\varphi: \mathbb{F}_2^N \rightarrow \mathbb{F}_2^\infty$ has property that $\varphi(x+y) - \varphi(x) - \varphi(y) \in S$ for some finite set S . Then $\varphi = \psi + \eta$ where $\psi: \mathbb{F}_2^N \rightarrow \mathbb{F}_2^\infty$ is linear and $|\text{im}(\varphi)| \ll |S|^c$.

Ben Green's favorite open problem.