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9:06 AM

# Dirichlet Forms

$\varphi, \psi : G \to \mathbb{R}$

$G$ – finite group

$$\langle \varphi \psi \rangle = \sum_{x \in G} \varphi(x) \psi(x)$$

$$= |G| E_x[\varphi(x)\psi(x)]$$

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$P_\pi : \underbrace{\mathbb{R}\langle G \rangle}_{\text{real-valued functions on } G} \to \mathbb{R}\langle G \rangle$

$\pi$ – prob. distribution on $G$

$$[P_\pi \varphi](x) = \sum_{y \in G} \varphi(xy)\pi(y)$$

think of this as 1 step of R.W.

$$= E\left[\varphi(X_1) \mid X_0 = x\right]$$

Dirichlet form

**Def:** $\mathcal{E}(\varphi, \varphi) = \langle (I - P_\pi)\varphi, \varphi \rangle$

$$= |G|\, E\left[\left(\varphi(X_0) - \varphi(X_1)\right)\varphi(X_0)\right]$$

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If $Z_0, Z_1$ — identical real-valued random vars

then

$$E\left[(Z_0 - Z_1) Z_0\right] = \frac{1}{2} E\left[(Z_0 - Z_1)^2\right]$$

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So,

$$\mathcal{E}(\varphi, \varphi) = \frac{|G|}{2} E\left[\left(\varphi(X_0) - \varphi(X_1)\right)^2\right]$$

$$= \frac{1}{2} \sum_{x,y \in G} \left(\varphi(x) - \varphi(xy)\right)^2 \pi(y)$$

$X_0$ — uniform in $G$

$X_1 = X_0 \cdot h$, where $h \in_\pi G$

from $G$, according to $\pi$

$\Gamma = \Gamma(G,S)$ Cayley graph

$g \in G$ — vertices

$(g, gs)$ — edges, $g \in G$, $s \in S$

$\Upsilon = \{ \gamma(g) \ \forall g \in G \}$

$\uparrow$

path from 1 to $g$ in $\Gamma(G,S)$

$\forall g \quad g = s_1 s_2 \cdots$ decomposition corresponding to $\gamma(g)$

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$$N_\Upsilon(s, C) = \max_{x,y \in C} \mu_s(x^{-1}y) \qquad \begin{array}{l} s \in S \\ C \subseteq G \end{array}$$

$\uparrow$ any subset

$\mu_s(g) = \#$ time $s$ used in decomposition of $g$

Thm: (Diaconis, Saloff-Coste)

$C \subseteq G$, $\bar{C} = C \cup \partial C$ (the boundary in $\Gamma$)

$C \subseteq G$, $\overline{C} = C \cup \partial C$ (the boundary in $\Gamma$)

$d = \operatorname{diam}(\overline{C}) \triangleq$ max length of decomposition.

$\pi, \tilde{\pi} \rightarrow$ symm. prob. distrib. on $G$

meaning $\pi(x^{-1}) = \pi(x)$

$S \subseteq \operatorname{support}(\pi) = \{g \in G \mid \pi(g) > 0\}$

<u>then</u>

$$\mathcal{E}_{\pi}(\varphi, \varphi) \geq \frac{1}{A} \mathcal{E}_{\tilde{\pi}}(\varphi, \varphi)$$

<u>where</u> $$A = d \max_{s \in S} \frac{N_r(s, \overline{C})}{\pi(s)}$$

Proof:

$$\varphi(x)-\varphi(xy) = \left[\varphi(x)-\varphi(xs_1)\right]+$$

$$+\left[\varphi(xs_1)-\varphi(xs_1s_2)\right]+\cdots$$

$$\cdots+\left[\varphi(xs_1\ldots s_{\ell-1})-\varphi(x\underbrace{s_1\ldots s_\ell}_{y})\right]$$

$$\left(\varphi(x)-\varphi(xy)\right)^2 \leq \ell\sum\left(\varphi(xs_1\ldots s_i)-\varphi(xs_1\ldots s_{i+1})\right)^2$$

also $\ell \leq d = \mathrm{diam}(\overline{C})$ Hw!

$$\sum_{x\in G}\left(\varphi(x)-\varphi(xy)\right)^2 \leq d\sum_{\substack{z\in G\\ s\in S}} N_\gamma(s,\overline{C})\cdot\left(\varphi(z)-\varphi(zs)\right)$$

$$\sum_{\substack{x\in G\\ y\in G}}\left(\varphi(x)-\varphi(xy)\right)^2\tilde{\pi}(y) \leq d\sum N_\gamma(s,\overline{C})\cdot \text{—}\|\text{—}$$

$$< A \sum \left( \varphi(z) - \varphi(zs) \right) \pi(s)$$

$C \overset{?}{=} \operatorname{supp}(\tilde{\pi})$