9/25 Ben Green

Prop (Inverse theorem for the U^2 norm)

Suppose f is 1-bounded function $[N] \to \mathbb{C}$ with $\|f\|_{U^2} > \delta$

Then $\exists\, \theta \in \mathbb{R}/\mathbb{Z}$ s.t. $\left| \mathbb{E}_{n \in [N]} f(n)\, e(-\theta n) \right| \geq c\delta^2$.

very lucky connection

Pf: We have $\|f\|_{U^2}^4 = \|f * f\|_2^2$

Hence by Parseval and convolution id, $\|f\|_{U^2} = \|\hat{f}\|_4$

Hence if $\|f\|_{U^2} > \delta$, then $\|\hat{f}\|_4^4 > \delta^4$

But by (a trivial instance of) Hölder's ineq. $\|\hat{f}\|_4^4 \leq \|\hat{f}\|_2^2 \|\hat{f}\|_\infty^2$

and, by Parseval again, and 1-boundedness of f, $\|\hat{f}\|_2 = \|f\|_2 \leq 1$

Putting all together: $\|\hat{f}\|_\infty > \delta^2$

This means $\exists\, r \in \mathbb{Z}/N'\mathbb{Z}$ s.t. $\left| E_{x \in G} f(x) e\left(\frac{-rx}{N'}\right) \right| \geq \delta^2$.

Take $\theta = \frac{r}{N'}$. ∎

Digression:

Equivalent form: (up to the δ-dependence)

Prop B: Suppose $f: [N] \to \mathbb{C}$ is a 1-bounded function with $\|\hat{f}\|_{U^{2}} \geq \delta$.

Then there is a connected 1-step nilpotent Lie group G of dimension $O_\delta(1)$, a discrete, compact subgroup $\Gamma \leq G$, a function $F: G/\Gamma \to \mathbb{C}$ with $\|F\|_{Lip} = O_\delta(1)$ and elements $g \in G$, $x \in G/\Gamma$ such that

$$\left| \mathbb{E}_n f(n) F(g^n x\Gamma) \right| \gg_\delta 1$$

(Prop A) $\Rightarrow$ (Prop B) trivial. Indeed, if $\|f\|_{U^2} \geq \delta$ then

$\left| \mathbb{E}_{n \in [N]} f(n) e(-\theta n) \right| \gg_\delta 1$ for some $\theta \in \mathbb{R}/\mathbb{Z}$.

Take $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$, $g \in \theta$

$F: \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is $F(t) = e(t) = e^{2\pi i t}$

(Prop B) $\Rightarrow$ (Prop A): Basic harmonic analysis:

Sketchy: Any Lipschitz function $F: (\mathbb{R}/\mathbb{Z})^d \to \mathbb{C}$ can be written as a Fourier series:

$$F(\vec{t}) = \sum_{m = O_\varepsilon(1)}^{O_\varepsilon(1)} c_m e(m \cdot t) + O(\varepsilon).$$

book: Zygmund's "Trigonometric series" covers all that harmonic analysis

Back to Roth's Theorem

Putting together: generalized von Neumann theorem and inverse theorem for Gowers U^2 norm; one obtains:

Lemma: Let $A \subseteq [N]$, $|A| = \alpha N$, $N > C\alpha^{-C}$

Suppose A has fewer than $\frac{1}{10}\alpha^3 N^2$ 3-term APs.
then $\exists$ a $\theta \in \mathbb{R}/\mathbb{Z}$ such that

$$\left| \mathop{\mathbb{E}}_{n \in [N]} f(n) e(-\theta n) \right| \geq c\alpha^6$$

here $f(n) = 1_A(n) - \alpha$. Call this "Linear bias".

(Remaining task is to remove $e(\theta n)$ and $|\cdot|$ signs, at the expense of replacing $[N]$ by a progression P of length $\geq N^{1/3}$.)

Lemma (Dirichlet's principle of the pigeons):
Let $\eta \in (0,1]$ and $\theta \in \mathbb{R}/\mathbb{Z}$. Then $\exists\, d \in \mathbb{Z}$, $0 < d < \frac{1}{\eta}$ such that $\|\theta d\|_{\mathbb{R}/\mathbb{Z}} < \eta$. (Here, $\|x\|_{\mathbb{R}/\mathbb{Z}}$ is the distance ~~of~~ from x to 0 in $\mathbb{R}/\mathbb{Z}$ also $|\{x\}|$.)

Pf: Look at the numbers $0, \theta, 2\theta, 3\theta, \ldots, \lfloor \frac{1}{\eta} \rfloor \theta \pmod 1$.

By the pigeonhole, there are distinct i and j, $0 \le i, j < \lfloor \frac{1}{\eta} \rfloor$

such that $\|j\theta - i\theta\|_{\mathbb{R}/\mathbb{Z}} \le \eta$. Take $d = j - i$. ∎

Corollary: Let $\theta \in \mathbb{R}/\mathbb{Z}$ and suppose $\delta \in (0,1)$.

Suppose $N > C\delta^{-C}$. Then I can partition $[N]$ into progressions $P_1 \cup \ldots \cup P_m$, each of length at least $N^{1/3}$, such that $\sup_{x, x' \in P_j} |e(\theta x) - e(\theta x')| \le \delta$ for all j.

Pf: Take $\eta = \frac{1}{20} \delta N^{-1/3}$ in Dirichlet's Lemma. Let d, $0 < d \le \frac{1}{\eta}$, be such that $\|\theta d\|_{\mathbb{R}/\mathbb{Z}} \le \eta$. Let P be any progression with common difference d and length $\le 2N^{1/3}$. Then

$$\sup_{x, x' \in P} |e(\theta x) - e(\theta x')| \le 2N^{1/3} |e(\theta d) - 1|$$

But $|e(t) - 1| = |2\sin(\pi t)| \le 2\pi \|t\|_{\mathbb{R}/\mathbb{Z}}$.

So this is $\le 2N^{1/3} \cdot 2\pi \|\theta d\|_{\mathbb{R}/\mathbb{Z}} \le \delta$.

But if $N > C\delta^{-C}$, it is "clear" that $[N]$ can be decomposed into progressions P with common difference d, length $\in [N^{1/3}, 2N^{1/3}]$.

Recall that $\left|\mathbb{E}_{n\in[N']} f(n)e(-\theta n)\right| \geq c\alpha^6$.

Take progressions $P_1 \ldots P_M$ as in the corollary, with $\delta = \frac{1}{2}c\alpha^6$.

Since $[N] = P_1 \cup \ldots \cup P_m$, we have

$$\sum_{j=1}^{m}\left|\sum_{n\in P_j} f(n)e(\theta n)\right| \geq c\alpha^6 \sum_{j=1}^{m}|P_j|$$

By triangle ineq, and the fact that $e(\theta n)$ varies by at most $\frac{1}{2}c\alpha^6$ on P_j. This implies:

$$\sum_{j=1}^{m}\left|\sum_{n\in P_j} f(n)\right| \geq \frac{1}{2}c\alpha^6\sum_{j=1}^{m}|P_j|$$

Note also that: $\sum_{j=1}^{m}\sum_{n\in P_j} f(n) = \sum_{n\in[N]} f(n) = 0$

+ add them together:

$$\sum_{j=1}^{m}\left(\sum_{n\in P_j} f(n)\right)_+ \geq \frac{1}{4}c\alpha^6\sum_{j=1}^{m}|P_j|$$

$$x_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise}\end{cases}$$

So there is at least one progression P_j such that

$$\sum_{n \in P_j} f(n) \geq \frac{1}{4} c\alpha^6 |P_j|.$$ That is: $$\frac{|A \cap P_j|}{|P_j|} \geq \alpha + c'\alpha^6.$$

This concludes Roth.

Behrend's example (1946):

There is a set $A \subseteq [N]$, $|A| \geq \frac{N}{e^{C\sqrt{\log N}}}$ with no 3-term APs.

Sketch: Let $L, d \geq 1$ be integer params. to be specified.

Let $S \subseteq \mathbb{R}^d$ be any set of points that lie on a sphere; as a subset of $\mathbb{R}^d$, S contains no 3APs. (due convexity)

(*) By pigeonhole principle, we may find such a set S with $S \subseteq \{1, \ldots, L\}^d$ and with $|S| \geq c\frac{L^{d-2}}{d}$.

(**) Map S to $\mathbb{Z}$ via map $\psi(x_1 \ldots x_d) = x_1 + (2L)^{2-1} x_2 + \ldots + (2L)^{d-1} x_d$.