B Green — 9/21

$AP_3(f_1 f_2 f_3) = \mathbb{E}_{x,d}\, f_1(x)\, f_2(x+d)\, f_3(x+2d)$

Roth — cont'd

$$f(x) = \begin{cases} 1_A(x) - \alpha 1_{[N]}(x) \\ 0 \quad \text{if } x \notin [N] \end{cases}$$

$\underbrace{\#\text{progressions in } A \;-\; \text{normalized } (\alpha) \cdot \#\text{ progressions in total}}_{\text{also, avg. \# progressions in sets of A's size}}$

<u>Lemma</u> if $A \subseteq [N]$, $|A| = \alpha N$, $N > C\alpha^{-c}$ Then either (i) A contains at least $\frac{1}{10}\alpha^3 N^2$ non-trivial 3APs and certainly ≥ 1

or (ii) $|AP_3(g_1 g_2 g_3)| \geq c\alpha^3$

where $g_1 g_2 g_3$ are 1-bounded ($|g_i(x)| \leq 1$)

and at least one of the g_i equals f.

$N' = 2N+1$

Pf: $AP_3(1_A 1_A 1_A) = \frac{1}{(N')^2}\#(3APs \text{ in } A)$

if (i) fails then, $\leq \frac{1}{10}\alpha\left(\frac{N}{N'}\right)^2 + \frac{\alpha N}{(N')^2} \leq \frac{2}{19}\alpha^3\left(\frac{N}{N'}\right)^2$

if $N > C\alpha^{-c}$

But $1_A = \alpha 1_{[N]} + f$ and AP_3 is trilinear so

$AP_3(1_A 1_A 1_A)$ as sum of 8 terms, one of which

$AP_3(\alpha 1_{[N]}, \alpha 1_{[N]}, \alpha 1_{[N]})$

$+ 7$ terms of form $AP_3(g_1 g_2 g_3)$.

But $AP_3(\alpha 1_{[N]}, \alpha 1_{[N]}, \alpha 1_{[N]}) = \alpha^3 AP_3(1_{N} 1_{N} 1_{N}) = \frac{\alpha^3}{(N')^2}\#\{3APs \text{ in } N\}$

$\geq \frac{\alpha^3}{(N')^2}\frac{1}{9}N^2$ .... use Δ-ineq ∎

So $|AP_3(f_1 f_2 f_3)| \geq \delta$ for 1-bounded $f_1 f_2 f_3 : G \to \mathbb{C}$

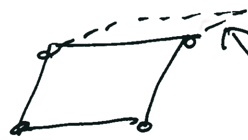
Definition: Gowers U^2-norm.

Suppose $f: G \to \mathbb{C}$ is a function.

Define $\|f\|_{U^2} := \left(\mathbb{E}_{x,h_1,h_2} \, f(x) \overline{f(x+h_1) f(x+h_2)} f(x+h_1+h_2) \right)^{1/4}$

Remarks: - Indeed norm, not entirely obv.

- Think $(x, x+h_1, x+h_2, x+h_1+h_2)$ as parallelogram.

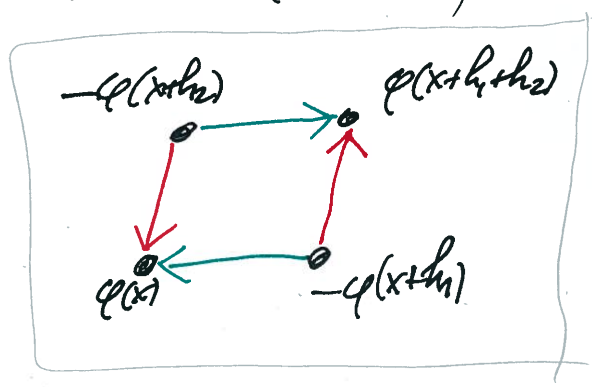


to generalize, go higher dim.

- when $f(x) = \begin{cases} e^{2\pi i \varphi(x)} \\ e(\varphi(x)) \end{cases}$ for some phase $\varphi : G \to \mathbb{R}/\mathbb{Z}$

then $\|f\|_{U^2} = \mathbb{E}_{x,h_1,h_2} \, e\big(\varphi(x) - \varphi(x+h_1) - \varphi(x+h_2) + \varphi(x+h_1+h_2)\big)$

$= \mathbb{E}_{x,h_1,h_2} \, e\big(\Delta_{h_1,h_2} \varphi(x)\big)$



Proposition: (Generalized von Neumann Theorem)

$f_1 f_2 f_3 : G \to \mathbb{C}$ are 1-bounded.

Then $|AP_3(f_1 f_2 f_3)| \leq \|f_i\|_{U^2}$, $i = 1, 2, 3$

Proof We use Cauchy–Schwarz in this form:

<u>if</u> X, Y are finite sets, $b : X \to \mathbb{C}$ is 1-bounded and $F : X \times Y \to \mathbb{C}$ arbitrary,

<u>then</u> $$\left| \mathbb{E}_{x \in X, y \in Y} b(x) F(x,y) \right|^2 \leq \mathbb{E}_{x \in X, y', y'' \in Y} F(x,y') \overline{F(x,y'')}$$

(pf exercise)

For von Neumann:

Apply Cauchy–Schwarz twice.

Sketch case $i=1$:

$$AP_3(f_1,f_2,f_3) = \mathbb{E}_{x,y} f_1(2x-y) f_2(x) f_3(y)$$

$$|\cdot|^2 \overset{C\text{-}S}{\leq} \mathbb{E}_x \mathbb{E}_{y,y'} f_1(2x-y) \overline{f_1(2x-y')} f_3(y) \overline{f_3(y')}$$

$$|\cdot|^4 \overset{C\text{-}S}{\leq} \mathbb{E}_{x x' y y'} f_1(2x-y) \overline{f_1(2x-y') f_1(2x'-y)} f_1(2x'-y')$$

$$= \|f\|_{U^2}^4 .$$

NB: Incidentally shown that quantity to be 4th rooted ≥ 0 and $\mathbb{R}$. ∎

Combining w/ preceding lemma, get:

<u>Lemma</u>: Let $A \subseteq [N]$ be a set, $|A| = \alpha N$, $N > C\alpha^{-C}$.

<u>Either</u> A has $\geq \frac{1}{10}\alpha^3 N^2$ 3-term APs

<u>Or</u> $\|f\|_{U^2} \geq c\alpha^3$ where $f = 1_A - \alpha 1_{[N]}$.

Key question: "Inverse question for Gowers norms"

What can I say when $\|f\|_{U^k} \geq \delta$?

(What is the rough structure of approximate these phases?)

why
what is

Discrete Fourier Transform:

$$G = \mathbb{Z}/N'\mathbb{Z} \qquad f: G \to \mathbb{C},\ r \in \underset{\left(\widehat{G}\right)}{\overset{}{\mathbb{Z}/N'\mathbb{Z}}} \text{ then}$$

def $$\widehat{f}(r) = \mathbb{E}_{x \in G} f(x) e\left(\frac{-rx}{N'}\right)$$

Properties:

(i) Plancherel thm: $$\langle f, g\rangle = \mathbb{E}_x f(x)\overline{g(x)} = \langle \widehat{f}, \widehat{g}\rangle$$

$$= \sum_r \widehat{f}(r)\overline{\widehat{g}(r)}$$

Parseval: $\|f\|_2 = \|\widehat{f}\|_2$

(ii) Inversion: $$f(x) = \sum_r \widehat{f}(r) e\left(\frac{rx}{N'}\right)$$

(iii) Convolution: $$f * g(x) := \mathbb{E}_t f(t) g(x-t)$$

then $$\widehat{f*g} = \widehat{f}\,\widehat{g}$$

$$\left(\text{Pf: exc: } \mathbb{E}_x e\left(\frac{rx}{N'}\right) = \begin{cases} 1, \text{ if } r=0 \\ 0, \text{ if } r \neq 0 \end{cases}\right)$$

Exercise: $$\mathbb{E}_x e\left(\frac{rx}{N'}\right) = \begin{cases} 1, \text{ if } r=0 \\ 0, \text{ if } r \neq 0 \end{cases}$$

Thm (Inverse Theorem for Gowers U^2-norm)

Let $f : [N] \to \mathbb{C}$ be 1-bounded and $\|f\|_{U^2} \geq \delta$.

Then $\exists \theta \in \mathbb{R}/\mathbb{Z}$ s.t. $|\mathbb{E}_x f(x) e(\theta x)| \geq c\delta^2$ ← i.e. "correlates" correlates with a linear phase.

Pf: Note $\|f\|_{U^2}^4 = \|f * f\|_2^2$ (ex.) Exercise: $\|f\|_{U^2}^4 = \|f * f\|_2^2$

$$= \|\hat{f}\|_4^4 .$$

So $\|f\|_{U^2} \geq \delta \Rightarrow \|\hat{f}\|_4^4 \geq \delta^4$

But $\|\hat{f}\|_4^4 \leq \|\hat{f}\|_2^2 \|\hat{f}\|_\infty^2$

$$= \|f\|_2^2 \|\hat{f}\|_\infty^2$$

$$\leq \|\hat{f}\|_\infty^2$$