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Mon+Fri 10am.

~def: $\underline{\approx \text{approx. group}}$: $\frac{|A\cdot A|}{|A|} \leq K$ approximation parameter

Structure of approx group?

(1) $A \subseteq \mathbb{Z}$: Freiman-Ruzsa Thm $\longleftrightarrow$ Szemerédi Thm for $K=4$

(2) $A \subseteq SL_2(\mathbb{F}_p)$ Helfgott: there are no approx subgroups which are not subgroups.

(3) Gromov Thm: Groups of polynomial growth.

nilpotent group $\overset{\approx}{\longleftrightarrow}$ poly growth

approx rings: $|A+A| + |A\cdot A| \leq K|A|$

Thm (Solymosi): No approximate subgroups of $\mathbb{C}$ w/ $|A| \geq N_0(K)$

Thm (Bourgain-Katz-Tao): No approx. subgr. of $\mathbb{F}_p$ with $N_0(K) \leq |A| \leq \frac{p}{N_0(K)}$

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Applications:

Exponential sum estimates over subgroups (Bourgain–Konyagin)

Notation: $d(n) = O_\varepsilon(n^\varepsilon)$
$\leftarrow$ # divisors of n.

§1: Roth's theorem: (Szemeredi, progression 3) 1953

"On certain sets of integers"

Thm: Suppose $A \subseteq [N]$ with $|A| \geq cN(\log\log N)^{-1/5}$,

then A contains 3 distinct elements in arithmetic progression.

Rem: (i) Weaker than Roth, he got $|A| \geq cN(\log\log N)^{-1}$ which is better.

(ii) Bourgain shown true if $|A| \geq N(\log N)^{-2/3+\varepsilon}$

Erdős (#3000): If $A \subseteq \mathbb{N}$ ~~and~~ has $\sum_{a\in A} \frac{1}{a} = \infty$ then A contains 3-term progression.

$\langle \approx \rangle$ Roth's $|A| \geq N(\log N)^{-1}$

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Density increment strategy:

Key to argument is dichotomy:

Proposition: Suppose $0<\alpha<1$. and $N > C\alpha^{-C}$.

Let $A \subseteq \cancel{[N]}\, P$ have size $\alpha[N]$,

P is progression of length N.

Then at least one of following true:

(i) A containts at least $\frac{1}{10}\alpha^3 N^2$ non-trivial 3-term progr. and in partic. at least 1.

(ii) There is progr. P', $|P'| \geq N^{1/3}$ such that

$$|A \cap P'| \geq (\alpha + c\alpha^6)\,|P'|$$

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Proof that Prop $\Rightarrow$ Roth

Suppose $A \subseteq [N]$, $|A| = \alpha N$ and no 3-term progr.

Attempt to apply iteratively.

$P_0 = [N]$, $\alpha_0 = \alpha$. If this remains valid,

I'll get $P_1 P_2 \dots$ and $\alpha_1 \alpha_2 \dots$

with property that $|P_{i+1}| \geq |P_i|^{1/3}$

and $\alpha_{i+1} \geq \alpha_i + c\alpha_i^6$ and $\frac{|A \cap P_i|}{|P_i|} \geq \alpha_i$

Easy ex: after $\frac{c}{\alpha^5}$ steps, $\alpha_i > 1$. ■

$\longrightarrow |P_i| \lesssim C\alpha^{-c} < C\alpha^{-c}$ for some $i \leq \frac{c}{\alpha^5}$

But clearly $|P_i| \geq N^{(1/3)^i}$ and hence $N^{(1/3)^{c/\alpha^5}} \leq C\alpha^{-c}$.

Taking a couple logs give $\alpha < c(\log\log N)^{-1/5}$. ■

Proof of prop:

In proving proposition, assume $P = [N]$ (else rescale)

Set $N' = 2N+1$

and write $G = \mathbb{Z}/N'\mathbb{Z}$, $[N] \subseteq G$ obvious way

If $f : [N] \to \mathbb{C}$ then $f : G \to \mathbb{C}$

by setting $f(x) = 0$ for $x = N+1, \dots, 2N+1$.

Def: Suppose $f_1, f_2, f_3 : G \to \mathbb{C}$

Define: $AP_3(f_1, f_2, f_3) = \mathbb{E}_{x,d \in G}\, f_1(x) f_2(x+d) f_3(x+2d)$

Note that $AP_3(1_A, 1_A, 1_A) = \frac{1}{N'^2} \times$ (# 3-term progressions in A including trivial ones. x,x,x)

where $1_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$

Will compare this quantity to: $AP_3(\alpha 1_{[N]}, \alpha 1_{[N]}, \alpha 1_{[N]})$

$$= \alpha^3 AP_3(1_{[N]}, 1_{[N]}, 1_{[N]})$$

In so doing we shall use the balanced function $f = 1_A - \alpha 1_{[N]}$

(Stop).