18.319 – Geometric Combinatorics with Prof. Igor Pak
Problem Set 1, Petar Maymounkov
In collaboration with Oren Weimann and Benjamin Rossman

Problem 1.

We consider a reformulation of the problem whereby the goal is to cover all points of the lattice $\mathbb{Z}^d$ with subsets of the form

$$P_v = \Big\{ v + \delta : v, \delta \in \mathbb{Z}^d \text{ and } \|\delta\|_1 \in \{0,1\} \Big\},$$

such that no two such subsets intersect. Let's call these subsets "tiles." Our answers to (a) and (b) will be in the form of sublattices of $\mathbb{Z}^d$ specifying the centers of the tiles. A sublattice of $\mathbb{Z}^d$ will be defined by a collection of basis vectors $b_1, \ldots, b_k$ such that the points of the sublattice are all linear combinations of the basis vectors over the ring $\mathbb{Z}$.

When $d = 2$ the tiling is given by:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

When $d = 3$ the tiling is given by:

$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

To prove that these tilings work, one checks that all lattice points are at ℓ_1 distance at least 3 apart from each other, and further that any point is at ℓ_1 distance at most 1 from a lattice point.

Problem 2a.

Claim 1. *We can draw an arbitrary line segment AB inside any triangulated polyhedron and preserve the triangulation by introducing some line segments between existing triangulation vertices and points on AB.*

Proof of Claim. Indeed, consider the region of AB crossing through the inside of a triangle Δ (of the initial triangulation). There are three cases:

1. AB crosses the boundary $\partial\Delta$ at two points, both of which are on the interior of the sides. Then AB splits Δ into a rectangle and a triangle. The former can be triangulated by introducing a single line segment.

2. AB crosses Δ in at least two points, one of which is on a vertex. No modifications are required in this case.

3. AB crosses $\partial\Delta$ only once, i.e. either A or B is inside Δ. To resume triangulation, we connect all three vertices of Δ with the endpoint of AB inside Δ.

□

Claim 2. *Any triangulation of the triangle that has no internal vertices is connected to the empty triangulation on that triangle.*

Proof of Claim. This is proved in two steps. Step 1 we show that every such triangulation is connected to the following "canonical" triangulation. Consider points A,B and C, and points a, b and c, respectively, in BC,AC and AB. And a triangulation of ABC consisting of the edges ab, bc and ca. Degenerate locations of a,b and c are allowed. They correspond to having one or more of these points coincide with A,B or C. Step 2 is to show that the canonical triangulation is connected to the empty one (this is trivial so we don't discuss it).

Step 1 is also proved trivially by induction. Consider the smallest triangle APQ with $P \in AC$ and $Q \in AB$. Let PQR be the smallest triangle that shares PQ with APQ. If $R \in AB$ or $R \in AC$ can merge the two triangles and repeat. Otherwise, $R \in CB$. Apply the same argument to all three vertices A, B and C. □

Proof of Problem. We are going to prove that every triangulation of ABC is connected to the empty one by inducting on the number of *internal* vertices. Our inductive hypothesis is that all triangulations with no more than $n-1$ internal vertices are connected to the empty triangulation. The base step is our previous claim.

Given a triangulation with n internal vertices, proceed as follows:

1. Pick an arbitrary internal vertex X and cut ABC with the line CX, which crosses AB in Y. Apply our first claim to make sure that the new figure is still triangulated. Notice that the number of internal vertices of ACY and BCY is $\leq n-1$
2. Apply inductive hypothesis to ACY and BCY to make them empty
3. Merge ACY and BCY.

□

Problem 2b.

Pick a vertex of the polygon A and connect all other vertices to A, thereby triangulating the polygon. Call this the "master" triangulation. Impose the master triangulation on top of the given one and use the claim from the previous section to make sure that the picture can be refined so that it is a triangulation. Apply the previous part of the problem to show that we can "empty" the triangles of the master triangulation. And done. Thus we have shown that all triangulations of the polyhedron are connected to the master triangulation.

Problem 3a.

We are going to describe a recipe for cooking a round convex body. Then we will show that all convex bodies can be cooked using this recipe. The recipe:

- Let the *band* be the set $B = \{(x,y) \in \mathbb{R}^2 : |y| \le 1/2\}$
- Let a *rotated band* be the set B_θ achieved by rotating B by θ around the origin
- Let a *rotated and shifted band* be the set $B_{\theta,\delta}$ achieved from B_θ after translation by δ units in the direction of $e^{i(\theta+\pi/2)}$
- We construct a round body using the following (infinite) algorithm:

  i. $C = B_{0,0} \cap B_{\pi/2,0}$ and $i = 1$
  ii. Loop:
  iii. Set $\theta_i = \pi \frac{2\left(i - 2^{\lfloor \log_2 i \rfloor}\right)+1}{2^{\lfloor \log_2 i \rfloor + 1}}$
  iv. Choose a displacement δ_i, so that $B_{\theta_i,\delta_i} \cap C$ has projection onto $e^{i(\theta+\pi/2)}$ of length 1. Such a choice is always available as we discuss later
  v. Set $C = C \cap B_{\theta_i,\delta_i}$ and $i = i + 1$
  vi. Repeat loop

The sequence $(\delta_1, \delta_2, \dots)$ describes the round body R through $R = \bigcap_{i \in \mathbb{N}^+} B_{\theta_i,\delta_i}$. This is also a unique representation up to rotation or reflection. Trivially, given a round body one can find a corresponding sequence.

More interestingly, though, this algorithm has the following property: If you have chosen the first k parameters δ_i, for $i \in [k]$, there is a continuum of allowable choices for δ_{k+1}. (Proof omitted.)

This observation gives us a trivial way of building an infinite family of different round bodies, simply by picking any non-constant infinite sequence of values for δ_2 in the allowable interval $[1/2 - \sqrt{2}/2, \sqrt{2}/2 - 1/2]$.

Problem 3b.

For any sequence $R = (\delta_1, \delta_2, \dots)$, one can look at the truncation of this sequence to the first k elements, call it R_k. R_k defines a polytope, and the sequence of perimeters of $R_1, R_2, \dots$ converges to the perimeter of R.

Let $R = (\delta_1, \delta_2, \dots)$ be the definition of a round body R. And let $U = (\delta_1^*, \delta_2^*, \dots)$ be the definition of the unit circle U. We are going to make a simple observation. In particular, the sequence R_k can be continuously transformed into U_k such that the derivative of the perimeter (of the polytope represented by the transforming sequence) remains 0. This can be shown using only classical geometry. (Proof omitted.) This means that the perimeter of R_k is equal to the perimeter of U_k and hence the two sequences converge to the same perimeter.

Problem 4.

Consider the face F that is closest to z.

i. If z's projection onto the hyperplane spanned by F falls onto the interior of F, we are done.

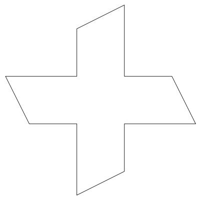

Figure 1: Counter-example picture.

ii. If it falls onto a boundary of F, say at a point Q, consider any other face F' whose boundary contains Q. Let W be any interior point of F' and consider the triangle ZQW. Since the polytope is convex, the angle between F and F' must be strictly acute, therefore $\angle ZQW$ must also be strictly acute. Therefore there exists a point $V \in QW$ such that $|ZV| < |ZQ|$ which gives us a contradiction, because it suggests that F' is closer to z than F.

iii. Otherwise the projection ray zR from z to F's hyperplane doesn't touch F at all. Let R be the intersection of this ray with the hyperplane of F. Since by assumption zR does not intersect F, zR must intersect some other face F' at some point W. If R separates z and W on zR, then z and W are separated by F's hyperplane, therefore W is outside the polytope, which is not possible. Therefore W is in the interior of zR or coincides with R. The former case, implies that F' is closer to z than F which is a contradiction. The latter case, is also a contradiction because W would have to be part of ∂F which is against our initial assumption.

This proof works for any-dimensional polytope (not just 3). For a non-convex counterexample consider the three-dimensional polytope whose projection onto $0xy$, $0yz$ and $0zy$ is described by the boundary in Figure 1.

Problem 5.

Following the discussion from class, the maximum diameter of a smooth convex body is always achieved at two boundary points such that the boundary tangents at both points are parallel. Similarly, if we consider all cuts that are orthogonal to the tangent at one of the boundary points, the shortest such cut must have it that it is orthogonal to the tangents at both ends. (These can also easily be formalized using Lagrange multipliers.)

If the minimum and maximum diameter (defined as above) of a convex body are unequal, they provide two examples of special cuts. If they are equal, there are infinitely many such cuts. So, in particular, round bodies have infinitely many special cuts.