6.854 Advanced Algorithms **Petar Maymounkov**
Problem Set 3 (September 27, 2005)
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Problem 1. In addition to `Q.min`, `Q.high`, and `Q.low[]`, we augment the VeB structure with `Q.cmax`. Where unlike `Q.min`, `Q.cmax` simply keeps a copy of the maximum element, but the maximum element itself is stored in the unaugmented part of the structure as usual.

Due to our design choice for `Q.cmax`, `insert` and `delete-min` are only trivially modified:

insert(x,Q):

1. If `Q` is empty, set `Q.cmax := x`, proceed with usual `insert`,
2. Else if, `x > Q.max`, set `Q.cmax := x`, proceed with usual `insert`,
3. Else, proceed with usual `insert`.

This is adding 1 extra step per level, on the running time of `insert` which touches $\log b$ levels, therefore total running time rmains $O(\log b)$.

delete-min(Q):

1. Call usual `delete-min(Q)` (which internally recurses with the augmented `delete-min`), it returns element `x`.
2. If `x` was the last element, i.e. `x = Q.cmax`, reset `Q.cmax := NULL`,
3. Return `x`.

Same running time argument as above.

find(x,Q):

```
if Q.min = x
  return x
else if Q.low[x_h] not empty
  return find(x_l,Q.low[x_h])
else
  return NULL
```

Running time is $T(b) = 1 + T(b/2) = O(\log b)$.

successor(x,Q):

```
if Q.min = x
  y_h := Q.high.min
  y_l := Q.low[y_h].min
  return (y_h,y_l)
else if (y_l := successor(x_l, Q.low[x_h])) != NULL
  return (x_h,y_l)
else
  y_h = successor (x_h, Q.high)
  return (y_h, Q.low[y_h].min)
```

Running time is $T(b) = 1 + T(b/2) = O(\log b)$.

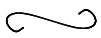
predecessor(x,Q):

```
if x = Q.min
  return NULL
```

```
else if (y_l := predecessor(x_l, Q.low[x_h])) != NULL
  return (x_h,y_l)
else
  y_h = predecessor(x_h,Q.high)
  return (y_h, Q.low[y_h].cmax)
```

Running time is $T(b) = 1 + T(b/2) = O(\log b)$.

Finally, we argue that the space of the augmented VeB is still ...



Problem 2. Note that VeBs support a regular `delete` operation for existing elements:

delete(x,Q):

```
if x = Q.min
  delete-min(x,Q)
else
  delete(x_l,Q.low[x_h])
  if Q.low[x_h] empty
    delete(x_h,Q.high)
```

Note that this operation still takes time $O(\log \log b)$ because if we fall in the scenario where the two `delete`'s are called, one of them is constant time. We are going to need the `delete` operation in order to implement `decrease-key`.

Initialize an array of n VeB's on $\{1, \ldots, C\}$ each. Keep a pointer to the first non-empty VeB.

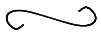
To **insert** an $x \in \{1, \ldots, nC\}$, let x_0 be the integer formed by the high order $\log n$ bits of x, and let x_1 be the integer formed by the remaining low order $\log C$ bits of x. Then insert x_1 into the x_0-th VeB.

To **decrease-key** x to y, first, `delete` x_1 from the x_0-th VeB and insert y_1 into the y_0-th VeB.

To **delete-min**, run `delete-min` on the VeB at the current pointer location, and if it is empty increase the pointer to point to the next VeB, and try again.

Observe that due to Dijkstra's monotone queue access, at most two consequetive VeBs in the array can ever be non-empty. Since we are keeping a pointer to the lower one at all times, the worst time `delete-min` could take is $O(\log \log C)$.

All of the above queue operations now run in $O(\log \log C)$ time. Finally, as a matter of space optimization, we can use a wrap-around array of only 2 VeBs, instead of n.



Problem 3a. We begin be listing all real costs:

1. Insert:
1A. Traverse down until hit blob or heap
1B. Occasionaly need to insert into heap

2. Decrease-key:
2A. Travel some number of nodes down
2B. May have to insert in heap, if hit a heap instead of blob
2C. May have to do a heap decrease-key, if already in heap

3. Delete-min:
3A. Do delete-min from heap that contains minimum
3B. Scan up tree nodes to find next minimum
3C. Occasionally, have to make heap from blob (time linear in size of blob)

Now we match up all real costs with their potential counterparts.

The **insert** operation takes amortized $O(k+k\Delta/t+I(t))$: k accounts for 1A, 2A, and 3C; $k\Delta/t$ accounts for 3B, because when each of t nodes deposits Δ/t potential at a node, the node has Δ potential for a scan during 3B; and $I(t)$ accounts for 1B.

The **decrease-key** operation takes amortized $O(D(t)+I(t))$: $D(t)$ accounts for 2C; $I(t)$ accounts for 2B.

The **delete-min** operation takes amortized $O(X(t))$, which accounts for 3A.

The so described matching between charged (amortized) costs for operations and real costs fully covers the algorithm, which proves the claim.

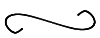
Problem 3b. Substituting $k = \sqrt{\log C}$, $t = 2^{\sqrt{logC}}$, $\Delta = C^{1/\sqrt{\log C}}$, $I(t) = O(1)$, $D(t) = O(1)$, and $X(t) = O(\log t)$ we get:

$$\text{Delete-min: } O(X(t)) = O(\log 2^{\sqrt{\log C}}) = O(\sqrt{\log C})$$

$$\text{Decrease-key: } O(D(t) + I(t) = O(1)$$

$$\text{Increase: } O(k + k\Delta/t + I(t)) = O\left(\sqrt{\log C} + \sqrt{\log C}\frac{C^{1/\sqrt{\log C}}}{2^{\sqrt{\log C}}} + 1\right) = O(\sqrt{\log C})$$

Therefore, we get Dijkstra's running time to be $O(m + n\sqrt{\log C})$.



Problem 4a. Let X_{ij} be the indicator random variable that the i-th inserted item collides with the j-th inserted item ($j < i$), where:

$$\Pr[X_{ij} = 1] \leqq \frac{1}{n^{1.5}}\frac{l}{n^{1.5}} + \frac{1}{n^{1.5}}\frac{i-l}{n^{1.5}} = \frac{i}{n^3}$$

This is the union bound of: (a) $h_1(x_i) = h_1(x_j)$ and $h_2(x_i)$ hitting an allocated spot, and (b) $h_2(x_i) = h_2(x_j)$ and $h_1(x_i)$ hitting an allocated spot.

Expected total number of collisions is now:

$$\begin{aligned} E\left[\sum_{i=1}^{n}\sum_{j=1}^{i-1} X_{i,j}\right] &= \sum_{i=1}^{n}\sum_{j=1}^{i-1} E\left[X_{i,j}\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{i-1} \frac{i}{n^3} \\ &\leqq \sum_{i=1}^{n} \frac{i^2}{n^3} \\ &\overset{n}{\to} \frac{1}{3} \end{aligned}$$

Problem 4b. Let's look at the terms comprising $\Pr[X_{ij} = 1]$ above: (a) $1/n^{1.5}$ uses that i and j are pairwise independent, and (b) $l/n^{1.5}$ is a union bound of l copies of $1/n^{1.5}$ which themselves assume only pairwise independence.

Note that it is also required that the two hash functions used are independent from each other, otherwise we wouldn't be able to multiply $1/n^{1.5}$ by $l/n^{1.5}$ e.g. above.

Therefore all we need is two independent hash functions that are each pairwise independant.

Problem 4c. Using Markov's inequality (or just the definition of expectation), with 1/2 probability, we get a pair of randomly selected pairwise

independant hash functions to produce no collisions. Therefore, we have to re-attempt to generate new hash functions expected 2 times until we get a perfect pair. Since the time for each attempt is $O(n)$, the expected running total running time is still $O(n)$.

Each hash function is described by 2 2-word seeds, therefore we need 8 words in total to describe both hash functions.